# The Wielandt ideal of a Lie algebra

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I hereby state that this thesis is my own work and that all sources used have been acknowledged.

Signed:

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# Abstract


By analogy with results in group theory, this thesis considers the Wielandt ideal, $\omega(L)$, of a Lie algebra, $L$. An exposition is given of results by Chevalley and Tuck from algebraic group theory which imply that, when $L$ is a finite-dimensional Lie algebra over a field of characteristic zero, $\omega(L)$ is a characteristic subgroup. This result, but more generally Tuck's, strengthens the connection between group theory and Lie algebra theory.

The Wielandt series and Wielandt length of a Lie algebra (also finite-dimensional over a field of characteristic zero) are then defined and results linking the Wielandt length to soluble length are proved. In the case of groups, the Fitting class of a soluble group and the nilpotent class of a nilpotent group also give interesting results in terms of the Wielandt length, but these do not turn out to be useful in the context of Lie algebras.

Stewart [17] characterised all Lie algebras of Wielandt length $1$. The final chapter of this thesis characterises all Lie algebras of Wielandt length $2$.

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# Contents

# Chapter 1

# Introduction

In group theory, given a group, $G$, the *Wielandt subgroup*, denoted $\omega(G)$, is defined to be the intersection of the normalisers of all subnormal subgroups. This subgroup reflects the subnormal structure of a group. Since an automorphism of $G$ permutes the subnormal subgroups of $G$, $\omega(G)$ is invariant under all automorphisms of $G$. Therefore, it is a characteristic subgroup of $G$. Thus, subnormal subgroups of $\omega(G)$ are subnormal in $G$ itself, and hence are normal in $\omega(G)$, by its definition. A group in which all subnormal subgroups are normal is called a $T$-group, and much is known about the structure of $T$-groups (see [15], p388-393).

The Wielandt subgroup was first studied by Wielandt, [20], in 1958. Wielandt proved that, if $G$ is a finite non-trivial group, then $\omega(G)$ is non-trivial. With this in mind, we inductively define the *Wielandt series* of the group by

$$\begin{aligned} \omega_1(G) &:= \omega(G) \quad \text{and, for } i \geq 1 \\ \omega_{i+1}(G)/\omega_i(G) &:= \omega(G/\omega_i(G)) \end{aligned}$$

Using Wielandt's result, for a finite group $G$, there is some $n$ such that $\omega_n(G) = G$. The least such $n$ for which this holds is called the *Wielandt length* of $G$.

There have been a number of investigations of the structure of groups in relation to the Wielandt series (for example, [4], [6] and also other works by Bryce, Cossey and Ormerod, like [5]. Also work by Casolo [7],[8] and by Brandl et.al. [3]). In particular, in [4], the relationship between the Wielandt length and the derived and Fitting lengths of finite soluble groups is investigated. One result obtained is that, if $G$ is a finite soluble group of Wielandt length $n$, then the derived length is no more than (approximately) $\frac{5}{3}n$ (depending upon the remainder of $n$ modulo 3, the bound may be slightly more than $\frac{5}{3}n$). Bryce and Cossey also investigate the relationship between the Wielandt length and the Fitting length. Their result in this case is that the Fitting length is at most one more than the Wielandt length for a finite soluble group.

There is a tradition of investigating analogies between group theory and the theory of Lie algebras. There are many surprising correspondences between these two fields. It is very often the case that analogous results will be true in groups and Lie algebras, while the methods of proof for the two cases may be very different. The interest comes in how often there are analogous results, even though the techniques involved are quite different. The book by Amayo and Stewart [1] presents a survey of results in infinite-dimensional Lie algebras which are parallel to results in infinite-groups. Stewart, [18] proved an analogue to a result of Wielandt [21]. In this work, we chart a parallel course to Wielandt, Camina, Bryce, Cossey, Ormerod, and the others mentioned, in the field of Lie algebras.

We investigate the analogues of all of the above group theory notions in the case of Lie algebras. We define the Wielandt subalgebra of a Lie algebra in a precisely analogous way. In Lie algebras, it is not obvious that the Wielandt subalgebra is an ideal, since an ideal must be invariant under inner derivations, rather than inner automorphisms. However, we prove a theorem of Tuck [19] which relies heavily on the theory of algebraic groups developed by Chevalley [9], which allows us to prove that, in a finite-dimensional Lie algebra over a field of characteristic 0, the Wielandt subalgebra is a characteristic ideal. Chapter 3 is dedicated to proving Tuck's result. Hartley [11] proved an analogue of Wielandt's result, that in finite-dimensional nontrivial Lie algebras over fields of characteristic zero, the Wielandt ideal is non-trivial. Given these two results, we can define the Wielandt series.

In Chapter 4, we consider the relationship between Wielandt length and other invariants of soluble Lie algebras. Because of Tuck's result, and many others which only hold when the underlying field is of characteristic 0, we will only work with Lie algebras over fields of characteristic 0. Bryce and Cossey's result about the Fitting length is perhaps their more important result. However, for Lie algebras, the relationship between Wielandt length and Fitting length is not interesting, since all soluble Lie algebras over fields of characteristic 0 are nilpotent-by-abelian, and hence have Fitting length at most 2. We next consider the relationship between the Wielandt length and the nilpotency class in the case of nilpotent Lie algebras. It turns out that, if $L$ is a nilpotent Lie algebra (over a field of characteristic 0), then $\omega(L) = \zeta(L)$, the centre of $L$. Thus, the Wielandt series and the upper central series coincide in this case.

Next, we investigate the relationship between the Wielandt length and the derived length of soluble Lie algebras. In the spirit of Bryce and Cossey, we obtain a bound on the derived length in terms of the Wielandt length. This bound is a significant improvement on the case for groups (where the derived length depends linearly on the Wielandt length) as we find a bound on the derived length which is of order the log of the Wielandt length. It is not known whether the bound obtained is the best possible, except in the case of Wielandt length 2.

We denote by $\mathfrak{W}_n$ the class of soluble Lie algebras over a field of characteristic 0 which have Wielandt length at most $n$. In chapter 5, we investigate the class $\mathfrak{W}_2$. In the course of characterising these algebras a question regarding matrices arises which is interesting in its own right. We prove that one of the possible classes of algebras in $\mathfrak{W}_2$ are characterised by a set of square matrices which commute with each other and such that no linear combination of these matrices has an eigenvalue in the underlying field. This surprising result means that the characterisation of $\mathfrak{W}_2$ depends very strongly on the underlying field. For example, if the field is algebraically closed, then there are no matrices without eigenvalues. We begin an investigation of these matrices, but this seems to be a very deep question. However, except for the open questions about these matrices, we completely characterise the algebras in $\mathfrak{W}_2$.

## 1.1 Notation

We use the symbol ■ to mark the end of a proof, or the end of a theorem, corollary or lemma if there is no proof.

We use $F$ to indicate an arbitrary field, $\mathbb{R}$ for the field of real numbers, $\mathbb{C}$ for the field of complex numbers, $\mathbb{Q}$ for the field of rational numbers and $\mathbb{Z}$ for the ring of integers.

If $X$ is a set, we use the notation $\langle X \rangle_F$ to denote the $F$-span of the set $X$ over a field $F$. Usually the field $F$ will be understood, and we will simply write $\langle X \rangle$.

We use the symbol $\leq$ to mean subalgebra, the symbol $\trianglelefteq$ to mean ideal and the symbol $\subseteq$ to mean subset.

# Chapter 2

# Wielandt notions in Lie algebras

## 2.1 Lie algebras and Preliminary notions

**Definition 2.1.1** *Let $F$ be a field. A* Lie algebra over *$F$ is an $F$-vector space, $L$ with a bilinear multiplication $L \times L \to L$, $(x, y) \mapsto [x, y]$, which satisfies the following properties:*

$$[x, x] = 0 \quad \text{for all } x \in L \tag{2.1}$$
$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \quad \text{for all } x, y, z \in L \tag{2.2}$$

Identity 2.2 is called the *Jacobi identity.*

If $H$ and $K$ are subspaces of a Lie algebra $L$, we define the set $[H, K]$ to be the subspace spanned by all products of the form $[h, k], h \in H, k \in K$.

We abuse notation often by writing $[H, x]$ when we mean $[H, \{x\}]$. It is clear that, for any subspaces $H$ and $K$ of $L$, and any $x \in L$, that $[H, K] = [K, H]$ and that $[H, x] = [x, H]$.

A subspace, $H$ of $L$ for which $[H, H] \subseteq H$ is called a *subalgebra.* An ideal of $L$ is a subalgebra, $H$, such that $[H, L] \subseteq H$. A Lie homomorphism (or just homomorphism) between Lie algebras $A$ and $B$ is a linear mapping from $A$ to $B$ which preserves the Lie product. An isomorphism is a bijective homomorphism and an automorphism is an isomorphism from a Lie algebra to itself.

Lie algebras are not associative. Therefore, we define a convention for writing repeated Lie products;

**Definition 2.1.2** *We define the* left-normed convention *for Lie products inductively,*

$$[x_1, \ldots, x_{n-1}, x_n] = [[x_1, \ldots, x_{n-1}], x_n].$$

*We also define the repeated product* $[x, ny]$ *for* $n$ *a positive integer. We define* $[x, 1y] = [x, y]$*, and for* $n \geq 1$ *we define* $[x, (n+1)y] = [[x, ny], y]$*. In the same way, if* $A, B, A_1, \ldots, A_n$ *are subspaces of* $L$*, then we define* $[A_1, \ldots, A_n]$ *and* $[A, mB]$*, for* $m$ *a positive integer.*

**Definition 2.1.3** *A subalgebra* $H$ *of* $L$ *is called a* subideal *if there is a sequence of subalgebras,* $H_0, \ldots, H_n$ *such that*

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{n-1} \trianglelefteq H_n = L.$$

*We write* $H \, si \, L$*.*

*The length of the shortest such series starting with* $H$ *is called the* defect of $H$.

**Definition 2.1.4** *The* centre *of a Lie algebra* $L$*, denoted* $\zeta(L)$*, is defined to be*

$$\zeta(L) := \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}$$

**Definition 2.1.5** *The* descending central series, $\{L^i\}$*, of a Lie algebra,* $L$*, is defined as follows:*

$$\begin{aligned} L^1 &:= L \quad \text{and for } i \geq 1, \\ L^{i+1} &:= [L^i, L] \end{aligned}$$

*We also define the* ascending central series, $\{\zeta_n(L)\}$*, of* $L$ *as follows,*

$$\begin{aligned} \zeta_1(L) &:= \zeta(L) \quad \text{and, for } i \geq 1, \\ \zeta_{i+1}(L) &:= \zeta\left(L/\zeta_i(L)\right) \end{aligned}$$

*Finally, we define the* derived series, $\{L^{(i)}\}$*, of* $L$ *as follows,*

$$\begin{aligned} L^{(0)} &:= L \quad \text{and for } i \geq 0, \\ L^{(i+1)} &:= [L^{(i)}, L^{(i)}] \end{aligned}$$

The ideal $[L, L] = L^2 = L^{(1)}$, is usually denoted $L'$ and is called the *derived subalgebra* of $L$.

Note that $L^i$, $\zeta_i(L)$ and $L^{(i)}$ are all characteristic ideals of $L$ for all $i \geq 1$.

**Definition 2.1.6** *A Lie algebra, $L$, is called* nilpotent *if there is some $n$ such that $L^{n+1} = 0$. If, furthermore, $L^n \neq 0$, then we say that $L$ is* nilpotent of class $n$.

*A Lie algebra is called* soluble *if there is some $n$ such that $L^{(n)} = 0$. If, furthermore, $L^{(n-1)} \neq 0$, then we say that $L$ is* soluble of derived length $n$.

It is easy to see (see [1],p7) that $L^n = 0$, if and only if $\zeta_{n-1}(L) = L$.

See [1], pp5-9 for a brief survey of the basic results of soluble and nilpotent Lie algebras. A result which is fundamental, and which we state without proof is,

**Theorem 2.1.7 ([1],p5)** *Let $L$ be a Lie algebra. Then*

$[L^m, L^n] \leq L^{m+n}$ *for all $m, n \geq 1$*

$L^{(n)} \leq L^{2^n}$ *for all $n \geq 0$*

*If $L$ is nilpotent of class $c$, then $L$ is soluble of derived length $\leq n$ where $n$ is the smallest integer $\geq log_2(c+1)$*

■

The next result is also fundamental, and we will use it freely without mention. The proof is exactly analogous to the corresponding one for group theory.

**Theorem 2.1.8** *If $L$ is a nilpotent Lie algebra, then all subalgebras are subideals.*

**Proof** : Suppose that $L^n = 0$, and $S \leq L$. Then,

$$S = S + L^n \trianglelefteq S + L^{n-1} \trianglelefteq \ldots \trianglelefteq S + L^2 \trianglelefteq S + L^1 = L.$$

■

**Definition 2.1.9** *If $H \leq L$, we define $H^\omega := \bigcap_{i \geq 1} H^i$.*

**Lemma 2.1.10 ([1], p11)** *If $H si L$, then $H^\omega \trianglelefteq L$.*

**Proof** : We first prove by induction that for any subalgebra $K$ of $L$, and for all $n \geq 1$,

$$[K, H^n] \leq [K, nH].$$

This is trivial for $n = 1$. If we assume that it is true for $n = k$ and all $K \leq L$, then

$$\begin{aligned}
[K, H^{k+1}] &= [K, [H^k, H]] \\
&\leq [H, K, H^k] + [K, H^k, H] \quad \text{by the Jacobi identity,} \\
&\leq [[H, K], kH] + [[K, kH], H] \quad \text{by induction,} \\
&= [K, (k+1)H] + [K, (k+1)H] \\
&= [K, (k+1)H].
\end{aligned}$$

This completes the induction. In particular, for all $n \geq 1$, $[L, H^n] \leq [L, nH]$. We now show that if

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_m = L$$

is a subideal of $L$, then,

$$[L, mH] \leq H.$$

Since $H \leq H_i$, for $1 \leq i \leq m$ and $H_{i-1} \trianglelefteq H_i$, $[H_i, H] \leq H_{i-1}$. Thus, since $L = H_m$, by induction $[L, mH] \leq H$. Then, for all $r \geq 0$,

$$\begin{aligned}
[L, H^{r+m}] &\leq [L, (r+m)H] \\
&= [[L, mH], rH] \\
&\leq [H, rH] \\
&= H^{r+1}.
\end{aligned}$$

Now, suppose that $x \in H^\omega$ and $y \in L$. Then, $x \in H^i$ for all $i \geq 0$. In particular, $x \in H^{m+i}$ for all $i \geq 1$. But then, by the above result,

$$[y, x] \in [L, H^{m+i}] \leq H^{i+1}$$

for all $i \geq 0$. Thus, $[y, x] \in H^\omega$, and $H^\omega \trianglelefteq L$. This completes the proof of the lemma. ■

We state without proof the following basic theorem of Lie algebras

**Theorem 2.1.11** *If $L$ is a finite-dimensional Lie algebra over a field of characteristic 0, then there exists a soluble ideal, $R_L$, called the* radical, *which contains all soluble ideals of $L$, and a nilpotent ideal, $N_L$, called the* nil radical, *which contains all nilpotent ideals.* ■

If $L$ is soluble then $R_L = L$ and if $L$ is nilpotent then $N_L = L$.

The following theorem, which we state without proof, is crucial to everything we are doing. We use it in almost every proof, and will generally use it without further comment. This result is not generally true in group theory, and it provides us with the power to obtain the results we do.

**Theorem 2.1.12 ([13], p51)** *Let $L$ be a finite-dimensional Lie algebra over a field of characteristic $0$, $R_L$ the radical of $L$, and $N_L$ the nil radical. Then $[R_L, L] \leq N_L$. In particular, if $L$ is soluble, then its derived subalgebra is nilpotent.* ■

**Definition 2.1.13** *A finite-dimensional Lie algebra over a field of characteristic $0$ for which $R_L = 0$ is called* semi-simple.

The following theorem is trivial to prove; a proof is in [13], p25.

**Theorem 2.1.14 ([13], p25)** *If $L$ is a finite-dimensional Lie algebra over a field of characteristic $0$, and $R_L$ is its radical, then $L/R_L$ is semi-simple.* ■

## 2.2 Derivations

**Definition 2.2.1** *Let $A$ be an $F$-algebra (Lie or associative). A linear map $D : A \mapsto A$ is called a* derivation *if*

$$D(xy) = (Dx)y + x(Dy)$$

*for all $x, y \in A$.*

**Lemma 2.2.2** *If $A$ is an $F$-algebra with a unit element, $1$, (which we associate with the unit element of $F$), and $D : A \mapsto A$ is a derivation of $A$, then $D(1) = 0$.*

**Proof** : We have

$$\begin{aligned} D(1) &= D(1.1) \\ &= D(1).1 + 1.D(1) \\ &= 2.D(1), \end{aligned}$$

which is to say that $D(1) = 0$. Thus, the lemma is proved. ■

If $A$ is an algebra with a unit element, then $F$ may be canonically embedded in $A$, by the mapping which maps $\alpha \in F$ to $\alpha.1 \in A$. All derivations are $F$-linear. so the above result implies that, under this embedding, $F$ is mapped to zero by all derivations.

**Lemma 2.2.3** *Suppose that $D_1$ and $D_2$ are derivations of $A$. Then the function $[D_1, D_2] := D_1D_2 - D_2D_1$ is also a derivation of $A$.*

**Proof** : It is clear that $[D_1, D_2]$ is a linear function on $A$. Thus, we need only show that it acts appropriately with respect to the multiplication of $A$. Well, if $x, y \in A$,

$$\begin{aligned}
[D_1, D_2](xy) &= D_1(D_2(xy)) - D_2(D_1(xy)) \\
&= D_1((D_2x)y + x(D_2y)) - D_2((D_1x)y + x(D_1y)) \\
&= D_1((D_2x)y) + D_1(x(D_2y)) - D_2((D_1x)y) - D_2(x(D_1y)) \\
&= (D_1(D_2x))y + (D_2x)(D_1y) + (D_1x)(D_2y) + x(D_1(D_2y)) \\
&\quad -(D_2(D_1x))y - (D_1x)(D_2y) - (D_2x)(D_1y) - x(D_2(D_1y)) \\
&= \Big((D_1(D_2x)) - (D_2(D_1x))\Big)y \\
&\quad +x\Big((D_1(D_2y)) - (D_2(D_1y))\Big) \\
&= \Big([D_1, D_2]x\Big)y + x\Big([D_1, D_2]y\Big)
\end{aligned}$$

as required. Thus the lemma is proved. ∎

**Lemma 2.2.4** *Let $L$ be a Lie algebra, and $x \in L$. The function $\varphi_x : L \to L$ defined by $\varphi_x(y) = [y, x]$ is a derivation called the* adjoint map associated *with $x$.*

**Proof** : The linearity of the Lie product means that $\varphi_x$ is obviously linear. For the multiplicative condition,

$$\begin{aligned}
\varphi_x([y, z]) &= [y, z, x] \\
&= [y, x, z] + [y, [z, x]], \quad \text{by the Jacobi identity,} \\
&= [\varphi_x(y), z] + [y, \varphi_x(z)]
\end{aligned}$$

as required. ∎

**Definition 2.2.5** *Derivations of the form $\varphi_x$ for $x \in L$ are called the* inner derivations *of $L$.*

In analogy with groups we can see that ideals of $L$ are the subspaces of $L$ which are closed under all inner derivations. In this spirit,

**Definition 2.2.6** *A subspace which is fixed by* all *derivations of $L$ is called a* characteristic ideal.

## 2.3 The Wielandt subalgebra

**Definition 2.3.1** *If $H \subseteq L$, then we define the* idealiser of $H$ in $L$, *denoted $I_L(H)$, to be the following set,*

$$I_L(H) := \{x \in L \mid [x, h] \in H \text{ for all } h \in H\}.$$

It is easy to prove that if $H$ is a subspace of $L$, then $I_L(H) \leq L$, and that $H \trianglelefteq I_L(H)$ (see [1], p6).

We now define the Wielandt subalgebra of a Lie algebra.

**Definition 2.3.2** *The* Wielandt subalgebra *of a Lie algebra $L$, denoted $\omega(L)$, is defined to be*

$$\omega(L) := \bigcap_{S \text{ si } L} I_L(S).$$

For any subideal, $S$, $I_L(S) \leq L$ and the intersection of subalgebras is a subalgebra. Therefore, $\omega(L) \leq L$. It is also clear that $\zeta(L) \leq \omega(L)$.

We will be looking exclusively at soluble Lie algebras. One reason for this is that solubility gives us much more structural information about a Lie algebra and allows us to obtain many more results than would otherwise be available. Another, and perhaps more compelling reason is that simple and semi-simple Lie algebras (the study of which accounts for much of the classical study of Lie algebras) do not yield any interesting information in the case of the Wielandt subalgebra. This is due to the following theorem.

**Theorem 2.3.3** *If $L$ is a semi-simple Lie algebra over a field of characteristic zero, then $\omega(L) = L$.*

**Proof** : Using Jacobson, [13], p71, we have that $L$ is the direct sum of simple ideals. But, in this case it is easy to see that any subideal of $L$ is, in fact, an ideal. Therefore, $\omega(L) = L$. ■

Thus, we restrict our study to soluble Lie algebras. Once this course is sufficiently far advanced, it may be interesting to look at how adding some semi-simple component to $L$ changes the Wielandt structure, but it seems prudent to consider only the soluble case at first.

Since subideals are permuted by automorphisms of $L$, it is clear that $\omega(L)$ is invariant under all automorphisms of $L$. In the case of groups (where the analogous result is also true), this is enough to show that the Wielandt subgroup is characteristic (and, in particular, normal). However, in Lie algebras, invariance under derivations is important, rather than automorphisms. So, it is not immediately obvious whether or not $\omega(L)$ is an ideal. In the next chapter, we prove a result of Tuck [19] which shows that, when the underlying field is of characteristic $0$, $\omega(L)$ is in fact a characteristic ideal.

However, we first prove a result, due to Hartley [11], which states that, if $L$ is a finite-dimensional Lie algebra over a field of characteristic zero, then all minimal ideals are contained in $\omega(L)$. In particular, this means that, in non-trivial finite-dimensional Lie algebras, $\omega(L)$ is non-trivial.

## 2.4 Hartley's result

First, some preliminary Lemmas. The first two follow results by Schenkman [16], while the last two follow results by Hartley [11].

**Lemma 2.4.1** *Let $L$ be a finite dimensional Lie algebra over a field of characteristic zero. The radical, $R$, and nil radical, $N$, of $L$ are invariant under every derivation, $\tilde{d}$ of $L$. Moreover, $\tilde{d}(R) \leq N$.*

**Proof** : Let $\tilde{d}$ be a derivation of $L$, and let $\tilde{L}$ be the split extension,

$$\tilde{L} = L \dot{+} \left\langle \tilde{d} \right\rangle,$$

where $[l, \tilde{d}] := \tilde{d}(l)$ for all $l \in L$.

Clearly $L \trianglelefteq \tilde{L}$. Let $\tilde{R}$ be the radical of $\tilde{L}$. Semi-simple algebras have no nontrivial soluble subideals. Thus, by taking the quotient of $\tilde{L}$ by $\tilde{R}$ and noting that $R si \tilde{L}$, we see that $R \leq \tilde{R}$. Thus, $R \leq \tilde{R} \cap L$. However, $\tilde{R} \cap L$ is clearly an ideal of $\tilde{L}$ and hence is an ideal of $L$. Therefore, $R = \tilde{R} \cap L$. Then, $R$ is an ideal of $\tilde{L}$ and so, in particular, $\tilde{d}(R) \leq R$. So $R$ is invariant under all derivations of $L$.

Let $R_1 = \left\langle R, \tilde{d} \right\rangle$. Since $\tilde{d}(R) \leq R$, $R_1$ is a subalgebra of $\tilde{L}$. Thus, $R_1$ is soluble, so $R_1'$ is nilpotent. Now, $R_1' \leq R$ and $R_1' \trianglelefteq R_1$, so $R_1' \trianglelefteq R$. If $N_R$ is the nil radical of $R$, then we have $R_1' \leq N_R$. If $x \in L$ and $\tilde{d}$ is the inner derivation associated with $x$, then $[R, x] = \tilde{d}(R) \leq R_1' \leq N_R$. By letting $x$ run through all of $L$, we obtain $[R, L] \leq N_R$. In particular, $[N_R, L] \leq N_R$. Thus, $N_R \trianglelefteq L$. Hence, $N_R \leq N$. However, $N \leq R$ and so $N \leq N_R$, yielding $N_R = N$. Therefore, $\tilde{d}(R) \leq N$. In particular, $\tilde{d}(N) \leq N$. This completes the proof of the lemma. ■

**Lemma 2.4.2** *Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero, and let $N$ be the nil radical of $L$. If $M \, si \, L$ is nilpotent, then $M \leq N$.*

**Proof** : Let $M \trianglelefteq M_1 \trianglelefteq \ldots \trianglelefteq M_n = L$. For $j = 1, \ldots, n$, let $N_j$ be the nil radical of $M_j$ and let $m_j \in M_j$. Then $M \leq N_1$ since $M$ is nilpotent. But $[M_j, m_{j+1}] \leq M_j$, so $m_{j+1}$ defines a derivation $\tilde{m}_{j+1}$ of $M_j$. By Lemma 2.4.1, $N_j \tilde{m}_{j+1} \leq N_j$ whence $N_j \trianglelefteq M_{j+1}$, since $m_{j+1}$ was arbitrary. Thus $N_j \leq N_{j+1}$. So we have

$$M \leq N_1 \leq \ldots \leq N_n = N,$$

and the proof of the lemma is complete. ■

**Lemma 2.4.3** *Let $L$ be a finite-dimensional nilpotent Lie algebra over a field of characteristic zero and let $M$ be a minimal ideal of $L$. Then $[M, L] = 0$.*

**Proof** : Suppose that the result is false. Then let $a \in L, b \in M$ be such that $[a, b] = c \neq 0$. Since $c \in M$, $c^L = M$. Thus,

$$b = \sum_i [c, x_{i_1}, \ldots, x_{i_k}]$$

for some $\{x_{i_j}\} \subseteq L$. Let $A$ be the subalgebra of $L$ generated by $a, b$ and the $\{x_{i_j}\}$ which occur in the above sum. Let $B = b^A$. Then $c = [a, b] \in [B, A]$ and so $b \in [B, A]$ by the expression for $b$. Thus, $[B, A] = B$ which means that $B = 0$, since $L$ is nilpotent. Therefore, $b = 0$, so $c = 0$ which contradicts our assumption that $c \neq 0$. Thus the lemma is proved. ■

**Lemma 2.4.4** *Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero. Let $B$ be a minimal ideal of $L$, and $N$ the nil radical of $L$. Then*

$$[B, N] = 0$$

*and, if $A$ is a nilpotent subideal of $L$, $[A, B] = 0$.*

**Proof** : If $B \not\leq N$, then $B \cap N = 0$, so $[B, N] = 0$. So we can assume, without loss of generality, that $B \leq N$. Let $N_1$ be a minimal ideal of $N$ contained in $B$. By lemma 2.4.3, $N_1 \leq Z(N)$, the centre of $N$. Thus, $B \cap Z(N) \neq 0$. However, $Z(N) \trianglelefteq L$ (since $N \trianglelefteq L$ and $Z(N)$ is characteristic in $N$), so $B \leq Z(N)$, which is to say $[B, N] = 0$.

Let $A si L$ be nilpotent. Then by Lemma 2.4.2, $A \leq N$, thus by the first part of this Lemma, $[A, B] = 0$. ■

**Theorem 2.4.5** *Let $L$ be a finite-dimensional Lie algebra, $A$ a subideal of $L$ and $B$ a minimal ideal of $L$. Then*

$$B \leq I_L(A)$$

**Proof** : We recall from Lemma 2.1.10 that $A^\omega \trianglelefteq L$. There exists some $n$ such that $A^n = A^\omega$, since $L$ is finite-dimensional. We form the quotient algebras $L/A^n$, $A/A^n$ and $(B + A^n)/A^n$, which we denote by $L^*$, $A^*$, and $B^*$ respectively. If $B^*$ is trivial, then $B \leq A^n \leq A$ and the result follows trivially. Thus, we can assume without loss of generality that $B^*$ is non-trivial and thus is a minimal ideal of $L^*$. Therefore, by Lemma 2.4.4, $[B^*, A^*] = 0$, which is to say that $[A, B] \leq A^n \leq A$. This completes the proof of the Lemma. ■

We now have the following immediate Corollary,

**Corollary 2.4.6 (Hartley)** *Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero, and let $B$ be a minimal ideal of $L$. Then $B \leq \omega(L)$. In particular, if $L$ is non-trivial, then $\omega(L) \neq 0$.*

**Proof** : By Theorem 2.4.5, $B$ idealises all subideals of $L$. Thus, by definition, $B \leq \omega(L)$. ■

# Chapter 3

# Algebraic Groups a la Chevalley

## 3.1 Introduction

In group theory, proving that the Wielandt subgroup is normal is trivial, since subnormal subgroups are permuted by automorphisms. Thus, the Wielandt subgroup is fixed by all automorphisms, and so is a characteristic subgroup. In Lie algebras, however, it is not so simple. The Wielandt subalgebra is still invariant under all automorphisms of the Lie algebra. However, this does not immediately imply that it is an ideal, since ideals must be fixed under inner derivations, rather than inner automorphisms. Despite this problem, all is not lost. For finite-dimensional Lie algebras over a field of characteristic zero, Tuck [19], proved that if a subalgebra is fixed by all automorphisms, then it is fixed by all derivations. Thus the Wielandt subalgebra is a characteristic ideal.

Tuck's result is extremely useful for all investigations of Lie algebras take a group theoretic approach. The benefit of this result is that any 'natural' characteristic subgroup in group theory will have an analogue in Lie theory. Of course the word 'natural' here is ambiguous, and what we mean is that the subgroup is invariant under automorphisms and does not use any particularly group-theoretic notions. Thus, the Frattini subgroup (the intersection of all proper maximal subgroups) has an analogue in Lie algebras, and it is characteristic (see [19] for a detailed exposition). The Wielandt subgroup is another example of such a subgroup.

This chapter is devoted to proving Tuck's result, also providing an exposition of Chevalley's results which are background to Tuck's. There is a very strong connection between group-theoretic results and results in Lie algebras. Tuck's result only served to make this connection stronger. Although the method of proof does not illuminate the reasons behind this connection to any great extent, it seems prudent to understand the proof since it is so fundamental to the work we pursue later.

Much of this exercise involves simply translating Chevalley's results into English. His work [9] has never been translated into English, and much of this chapter is simply a translation of his work into English. The task of constructing this chapter also involved pruning Chevalley's results. The final theorem needed from [9] appears on p179, and relies on most of the work in the preceding 170 pages. However, there is also much more in [9] which is useful to the general theory, but not explicitly to our purpose. Thus, a secondary task in constructing this chapter was to compress 180 pages of algebraic group theory into a manageable size, given that our purpose is really to prove just one theorem. Despite this, I believe that the task is worthwhile, since this theorem is of such overriding importance to the basics of the task at hand.

The fact that this chapter ended up more than 30 pages long indicates, I believe, that Tuck's result is far from trivial. Most of the work in Chevalley is now considered fairly standard algebraic group theory, but to prove the work in detail still clearly requires a lot of room.

The precise statement of Tuck's result, and the form in which we prove it is the following

**Theorem 3.1.1 (Tuck, [19])** *Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero, and suppose $G$ is the group of all (algebra) automorphisms of $L$. If $H$ is a subalgebra of $L$ which is invariant under $G$, then $H$ is a characteristic ideal of $L$.* ■

Tuck applied Theorem 3.1.1 to the Frattini subalgebra of a Lie algebra, and we apply it in the same way to the Wielandt subalgebra:

**Corollary 3.1.2** *If $L$ is a finite-dimensional Lie algebra over a field of characteristic zero, then $\omega(L) \trianglelefteq L$.*

**Proof** : We know $\omega(L)$ is a subalgebra of $L$ which is invariant under all (Lie algebra) automorphisms. Therefore, by Theorem 3.1.1 $\omega(L)$ is a characteristic ideal, and in particular an ideal. ■

We can only prove both this result and Hartley's, Corollary 2.4.6, when the underlying field is of characteristic zero. Without these results it is difficult to see how we can prove anything. Thus, we assume henceforth that the underlying field of all our Lie algebras is of characteristic zero.

We now move on to the work of Chevalley, [9], which is the background needed for proving Theorem 3.1.1.

Chevalley's work in [9] and [10], constituted the first comprehensive study of algebraic groups from a purely algebraic point of view. This work is fundamental in the field of algebraic groups. Although we follow many of the results of [9], we do so only with the purpose of proving Theorem 3.1.1, and so do not give justice to the results contained in [9].

When the underlying fields are $\mathbb{R}$ or $\mathbb{C}$, then algebraic groups are Lie groups. Using Lie groups would make our work easier, because the Lie algebra of a Lie group is the tangent space at the identity, and the Lie product arises from differentiation. However, in arbitrary fields of characteristic zero, we do not have convergence of Cauchy sequences. Because of this, we do not have notions such as differentiation or $C^\infty$ functions. To get around this, we use rational functions, which do map into the appropriate field and also have easily calculated derivatives. Much of this chapter involves defining the necessary machinery to define algebraic groups.

Although it would make our work much easier, it would not be satisfactory to restrict our investigations to Lie algebras over $\mathbb{R}$ and $\mathbb{C}$. In Chapter 5, we encounter a class of Lie algebras whose structure depends very strongly on the underlying field. Informally, the further the underlying field is away from being algebraically closed, the larger this class seems to be. Thus, over $\mathbb{C}$, which is algebraically closed, this class is empty, and over $\mathbb{R}$ this class is relatively small. We have been unable to completely characterise this class, due to the dependence on the underlying field, but certainly it seems prudent to consider the most general case for the underlying field.

Over this entire chapter, we assume that $F$ is a fixed field of characteristic $0$. If $V$ is a vector space over $F$, then $\mathfrak{E}$ will be the (associative) algebra of all linear maps from $V$ to $V$. $V^*$, the dual of $V$, is the set of all linear maps from $V$ to $F$.

## 3.2 Co-ordinate Functions

Let $V$ be a vector space over $F$, and $\mathfrak{E}$ be the set of all $F$-linear maps from $V$ into itself (the endomorphism ring of $V$). Also, let $\{v_1, \ldots, v_n\}$ be a basis for $V$.

Define the function $X_{ij} : V \to V$ by

$$X_{ij} v_k = \delta_{kj} v_i$$

where $\delta_{kj}$ is the Kronecker delta. If $s \in \mathfrak{E}$, then we can write

$$s = \sum_{i,j=1}^{n} u_{ij}(s) X_{ij},$$

where $u_{ij}(s) \in F$. We call the $u_{ij}(s)$ the *co-ordinates of $s$ with respect to the basis $\{v_1, \ldots, v_n\}$*. Having fixed our basis, we can associate $s$ with the matrix $\left(u_{ij}(s)\right)$. The $n^2$ functions $u_{ij} : \mathfrak{E} \to F$ are called a *system of co-ordinate functions* for $\mathfrak{E}$ .

If $L$ is an extension field of $F$, then define

$$V^L := L \otimes_F V.$$

We can associate $V^L$ with the space of linear combinations of elements of $V$ with coefficients in $L$.

Since $\{v_1, \ldots, v_n\}$ is an $L$-basis for $V^L$, the functions $X_{ij}$ form a basis for the space of endomorphisms of $V^L$. Therefore, we can identify the space of endomorphisms of $V^L$ with $\mathfrak{E}^L$ $(= L \otimes_F \mathfrak{E})$.

We can now see that the $u_{ij}$ uniquely extend to a system of co-ordinate functions for $\mathfrak{E}^L$, which we will also label $u_{ij}$.

**Definition 3.2.1** *Let $V$ be an $F$-space, and $L$ an extension field of $F$. If $u_{ij}$, $1 \leq i, j \leq n$, form a system of co-ordinate functions for $\mathfrak{E}$ (and so also for $\mathfrak{E}^L$), and $s \in \mathfrak{E}^L$, then we define the ring $F[s]$ to be the ring $F[u_{11}(s), u_{12}(s), \ldots, u_{nn}(s)]$. We define the field $F(s)$ to be the field of quotients of $F[s]$.*

## 3.3 The field of rational functions over $V$

**Definition 3.3.1** *Let $V$ be a vector space over $F$. Let $B$ be a basis for $V$, and $M$ the monoid freely generated by $B$ (the identity of this monoid will be considered as the empty string, but written as 1 and identified with the identity of $F$). Let $T$ be the vector space over $F$ with $M$ as a basis. Note that since $B \subseteq M$, it is clear that $V \subseteq T$. We make the obvious definition of multiplication in $T$. If $x, y$ are elements of $M$, then they multiply as in $M$ and we extend the multiplication to all of $T$ bilinearly. $T$ is called the* tensor algebra on $V$.

*Let $\mathfrak{I}$ be the ideal of $T$ generated by all elements of the form $b \otimes b' - b' \otimes b$ for $b, b' \in B$. The algebra $S := T/\mathfrak{I}$ is called the* symmetric algebra on $V$.

$S$ is a commutative algebra. To see this, any element of $T$ can be written as a sum of products of elements of $B$. Modulo $\mathfrak{I}$, the elements of $B$ commute pairwise. Thus, it is clear that $S$ is commutative. We write the product in $S$ as juxtaposition and ignore the fact that the elements are cosets. Thus, $x\mathfrak{I} \otimes y\mathfrak{I}$ is written as $xy$.

$S$ is also an integral domain, so that we can form the field of quotients of $S$ (see Herstein [12], p140), which we denote by $R$. The elements of $R$ are called *rational expressions* in the elements of $V$.

It is straightforward to see that $T$ is the free associative $F$-algebra, freely generated by $B$, and that $S$ is the free associative, commutative $F$-algebra, freely generated by $B$.

We form the symmetric algebra $S^*$ on $V^*$. Let $\mu$ be the algebra of all functions from $V$ to $F$. It is clear that $V^* \leq \mu$. This inclusion map is linear. Since $S^*$ is relatively free, we can extend the inclusion map to a homomorphism from $S^*$ to $\mu$, denoted by $\pi$. The ring $\pi(S^*)$ is called the *ring of polynomial functions on $V$*, and the elements are called *polynomial functions on $V$*. Since $F$ is infinite, $\pi$ has trivial kernel.

We denote by $R^*$ the field of rational expressions in the elements of $V^*$. We would like to define the notion of *rational functions* over $V$ in the same way as we did for polynomial functions. However, this does not work in the same way. The reason for this is that if $\lambda \in V^*$, then $\lambda$ has a multiplicative inverse in $R^*$, but not in the algebra of functions from $V$ to $F$ (since $\lambda(0) = 0$). We can partially fix this problem in the following way.

**Definition 3.3.2** *Let $r^*$ be an element of $R^*$, and $x \in V$. We say that $r^*$ is* defined at $x$ *if we can write $r^* = PQ^{-1}$, where $P, Q \in S^*$ and $(\pi Q)(x) \neq 0$.*

*If $r^*$ is defined at $x$, and $r^* = PQ^{-1}$, we say that $(\pi P)(x)\Big((\pi Q)(x)\Big)^{-1}$ is* the value of $r^*$ at $x$, *and we denote this by $r^*(x)$.*

The set of elements $r^* \in R^*$ which are defined at a point $x \in V$ form a subring of $R^*$ which contains $S^*$, and the function $r^* \mapsto r^*(x)$ is a homomorphism of this ring into $F$.

**Definition 3.3.3** *If $E \subseteq V$ and $H : E \to F$, then we say that $H$ is a* rational function on $V$ *if there is some $r^* \in R^*$ such that $E$ is the set of points of $V$ at which $r^*$ is defined and, for all $x \in E$, we have $H(x) = r^*(x)$.*

If $r^*$ is a rational expression on $V$, then it is easy to see that $r^*$ uniquely determines $H$. We write $\mathbf{R}$ for the set of rational functions over $V$.

We want to prove that there is a one-to-one correspondence between $R^*$ and $\mathbf{R}$. To do this, we need to show that two rational expressions over $V^*$ which give rise to the same rational function over $V$ are equal. For this we introduce the notion of an algebraically dense set.

**Definition 3.3.4** *A subset $E$ of a vector space $V$ over $F$ is called* algebraically dense *in $V$ if there exists a polynomial function $Q \neq 0$ over $V$ such that $E$ is the set of points of $V$ for which $Q$ takes a non-zero value.*

An algebraically dense set cannot be empty. Since $\pi$ has trivial kernel, any polynomial which is not identically zero takes a non-zero value on some element of $V$.

**Lemma 3.3.5** *The intersection of a finite number of algebraically dense sets is dense.*

**Proof** : If $E_1, \ldots, E_n$ are algebraically dense sets such that, for each $i$, $Q_i \neq 0$ is the polynomial which takes non-zero values on $E_i$, then the polynomial $Q = Q_1 \ldots Q_n \neq 0$ is a polynomial which takes non-zero values on $\cap E_i$. This completes the proof of the Lemma. ■

**Lemma 3.3.6** *Let $V$ be a vector space over $F$ and $V^*$ the dual of $V$. Let $E$ be an algebraically dense set in $V$. A rational expression $r^*$ in the elements of $V^*$ which is defined on $E$ and takes the value $0$ on $E$ is identically zero.*

**Proof** : Write $r^* = PQ^{-1}$, where $P, Q \in S^*$. Now, $(\pi Q)(x) \neq 0$ on $E$, so we have that $(\pi P)(x) = 0$ for all $x \in E$. Let $Q_1$ be a polynomial on $V$ such that $E$ is the set of points for which $(\pi Q_1) \neq 0$. Then we have that $(\pi P)(x)(\pi Q_1)(x) = 0$ for all $x \in V$. That is to say that $PQ_1 = 0$. But $Q_1 \neq 0$ since $E$ is not empty. Therefore, $P = 0$ since $S$ is an integral domain, whence $r^* = 0$. This completes the proof of the Lemma. ■

We have that any element $r^*$ of $R^*$ gives rise to an algebraically dense set (the set on which $r^*$ is defined), and then to a rational function on $V$. We have proved now that if $r^*$ and $s^*$ give rise to the same rational function on $V$, then they must be equal. If $H$ is the rational function which $r^*$ gives rise to, then the mapping $r^* \mapsto H$ of $R^*$ into $\mathbf{R}$ is a homomorphism. Therefore, $R^*$ and $\mathbf{R}$ are isomorphic. In particular, $\mathbf{R}$ is a field.

Thus we have extended the notion of polynomial functions to that of rational functions.

## 3.4 Derivations

The main result of this section is that any linear function from a vector space, $V$, to its symmetric algebra, $S$, may be extended to a derivation of the field of rational expressions over $V$, which we denoted by $R$. This result is Proposition 5, p24 of Chevalley [9]. We first prove some preliminary results from [9].

Recall that by Definition 2.2.1, if A is an $F$-algebra (Lie or associative), a linear map $D : A \mapsto A$ is called a *derivation* if

$$D(xy) = (Dx)y + x(Dy)$$

for all $x, y \in A$.

**Lemma 3.4.1** *Let $V$ be a vector space over $F$, $T$ the tensor algebra over $V$, and $E$ be the space of linear transformations of $T$. Let $\Psi$ and $\Phi$ be linear functions from $V$ into $E$ and let $u \in T$. Then, there exists a unique element $X$ of $E$, such that*

$$X(xy) = \Big(\Phi(x)\Big)(X(y)) + \Big(\Psi(x)\Big)(y)$$

*for all $x \in V$, and $y \in T$, and such that $X(1) = u$.*

**Proof** : Let $B$ be a basis for $V$. The monoid $M$ freely generated by elements of $B$ is a vector space basis for $T$. Every element of this monoid, except for 1, may thus be uniquely expressed in the form $b_1 \dots b_n$, where $n \geq 1$ and $b_1, \dots, b_n \in B$. For, $b_1, \dots, b_n \in M$, we inductively define the element $\xi(b_1, \dots, b_n)$ as follows,

$$\xi(1) = u$$

$$\xi(b) := (\Phi(b))(u) + (\Psi(b))(1)$$

for all $b \in B$, and supposing that $\xi(b_1, \dots, b_n)$ is already defined for all $b_1, \dots, b_n \in B$,

$$\xi(b_1, \dots, b_{n+1}) := (\Phi(b))(\xi(b_2, \dots, b_{n+1})) + (\Psi(b_1))(b_2 \dots b_{n+1}).$$

Since the elements of the form $b_1 \dots b_n$ and 1 together form a basis for $T$, the function $\xi$ may be uniquely extended to an element of $E$. Thus, there is a linear function $X : T \mapsto T$, such that $X(1) = u$ and $X(b_1 \dots b_n) = \xi(b_1, \dots, b_n)$ for all $b_1, \dots, b_n \in B$. Since $X$ is linear, it is clear from the definition of $\xi$ that $X$ satisfies the required properties.

We now prove the uniqueness of $X$. To do this, it is sufficient to prove that if $Y \in E$ is such that

$$\begin{aligned} Y(1) &= 0, \quad \text{and} \\ Y(xy) &= \Big(\Phi(x)\Big)(Y(y)), \end{aligned}$$

for all $x \in V$ and all $y \in T$, then $Y$ is the zero function. Let $T'$ be the kernel of $Y$. By assumption $1 \in T'$. $T'$ is a subspace of $T$. By the other condition on $Y$, if $x \in V$ and $y \in T'$, then $xy \in T'$. But, the elements of $V$ generate $T$ by the operation of left-multiplication and the vector space operations. Thus, $T' = T$. This proves that $Y$ is the zero function, and thus that $X$ is unique. This completes the proof of the lemma. ■

**Lemma 3.4.2** *Let $V$ be an $F$-space, and $T$ its tensor algebra. Suppose that $D_1 : V \mapsto T$ is a linear function. Then there exists a derivation $D : T \mapsto T$ which extends $D_1$.*

**Proof** : If $x \in V$, define $\Phi(x) : T \to T$ by $\Phi(x)y = xy$, and $\Psi(x) : T \to T$ by $\Psi(x)y = D_1(x)y$. By Lemma 3.4.1, there exists a linear function $D : T \mapsto T$ such that

$$\begin{aligned} D(xy) &= \Big(\Phi(x)\Big)(D(y)) + \Big(\Psi(x)\Big)(y) \\ &= xD(y) + D_1(x)y \end{aligned}$$

for all $x \in V$ and all $y \in T$, and such that $D(1) = 0$. If $x \in V$, we have

$$\begin{aligned} D(x) &= D(x.1) \\ &= xD(1) + D_1(x)1 \\ &= D_1(x), \end{aligned}$$

so $D$ extends $D_1$.

We now prove that $D$ is a derivation. It is sufficient to prove that if $x \in V$, $y, y' \in T$ and

$$D(y'y) = D(y')y + y'D(y)$$

then

$$D\Big((xy')y\Big) = D(xy')y + xy'D(y). \tag{3.1}$$

So, suppose that

$$D(y'y) = D(y')y + y'D(y).$$

Then

$$\begin{aligned} D((xy')y) &= D(x(y'y)), \quad \text{since } T \text{ is associative,} \\ &= xD(y'y) + D(x)y'y, \quad \text{by definition,} \\ &= x\Big(D(y')y + y'D(y)\Big) + D(x)y'y \\ &= \Big(xD(y') + D(x)y'\Big)y + xy'D(y) \\ &= D(xy')y + xy'D(y), \quad \text{as required.} \end{aligned}$$

Thus $D$ is a derivation. If a derivation is defined on a set of generators for $T$ as an algebra, then its linear structure and the multiplicative condition give its values on the rest of $T$. Therefore, $D$ is unique and the proof of the Lemma is complete. ■

We now prove the main result of this section;

**Theorem 3.4.3** *Let $V$ be a vector space, $S$ the symmetric algebra over $V$ and $R$ the algebra of rational expressions in the elements of $V$. Let $\varphi : V \mapsto S$ be linear. Then $\varphi$ may be uniquely extended to a derivation, $D_\varphi$, of $R$. $D_\varphi$ maps $S$ into itself.*

**Proof** : Let $T$ be the tensor algebra over $V$ and $\mathfrak{J}$ the ideal generated in $T$ by all elements of the form $x \otimes x' - x' \otimes x$ for $x, x' \in B$. Let $S_0$ be a subspace of $T$ which complements $\mathfrak{I}$ (as a vector space). The canonical homomorphism from $T$ to $S$ then induces a bijective linear function from $S_0$ to $S$. If $x \in V$, we let $\psi(x)$ denote the element of $S_0$ which belongs to the same coset as $\varphi(x)$, modulo $\mathfrak{I}$ . Now, $\psi : V \to T$ is linear, so, by Lemma 3.4.2, $\psi$ may be extended to a derivation, $\Delta$, of $T$.

If $x, y \in V$, then,

$$\Delta(x \otimes y - y \otimes x) \quad = \quad \Delta(x) \otimes y + x \otimes \Delta(y) - \Delta(y) \otimes x - y \otimes \Delta(x)$$

Since $S$ is commutative, $\Delta(x \otimes x' - x' \otimes x)$ maps to zero under the canonical homomorphism from $T$ to $S$. Therefore, $\Delta(x \otimes x' - x' \otimes x) \in \mathfrak{I}$, so $\Delta$ maps $\mathfrak{I}$ into itself. Thus, $\Delta$ induces a derivation $D^S_\varphi$ of $S$. It is clear that $D^S_\varphi$ extends $\varphi$. We now prove that we can extend $D^S_\varphi$ to a derivation of $R$. This is a general fact about extending a derivation of integral domains to a derivation of the field of quotients (see Kolchin, [14], p63-4). We adapt the proof of Kolchin to our purposes.

We consider derivations of fields for a minute. Suppose that $L$ is an extension field of $F$ and that $\delta$ is a derivation over $L$ (considering the ground field to be $F$). Then if $r, s \in L$ and $s \neq 0$, we have

$$\begin{aligned} \delta(r) &= \delta(rs^{-1}s) \\ &= \delta(rs^{-1})s + rs^{-1}\delta(s) \quad \text{whence} \\ \delta(rs^{-1}) &= \Big(\delta(r)s - r\delta(s)\Big)s^{-2} \end{aligned}$$

With this in mind, the following definition is the only possible one.

Let $\delta$ be a derivation of $S$ and suppose that $as^{-1} \in R$, with $a, s \in S$ and $s \neq 0$. We define

$$\delta(as^{-1}) := \Big((\delta a)s - a(\delta s)\Big)s^{-2}.$$

We want to prove that this makes $\delta$ a derivation of $R$. The first thing to prove is that this definition of $\delta$ is well-defined. If $s = 1$, so that $as^{-1} \in S$, then the two definitions of $\delta$ coincide (since $\delta(1) = 0$, by Lemma 2.2.2). We now prove that if $as^{-1} = a_1 s_1^{-1}$ then $\delta(as^{-1}) = \delta(a_1 s_1^{-1})$. That is to say that

$$\Big((\delta a)s - a(\delta s)\Big)s^{-2} = \Big((\delta a_1)s_1 - a_1(\delta s_1)\Big)s_1^{-2},$$

which is equivalent to

$$\Big((\delta a)s - a(\delta s)\Big)s_1^2 - \Big((\delta a_1)s_1 - a_1(\delta s_1)\Big)s^2 = 0.$$

Well,

$$\begin{aligned}
& (\delta a)ss_1^2 - a(\delta s)s_1^2 - (\delta a_1)s_1s^2 + a_1(\delta s_1)s^2 \\
= \; & (\delta a)s_1.ss_1 - a_1(\delta s)ss_1 - (\delta a_1)s.ss_1 + a(\delta s_1)ss_1, \quad \text{since } as_1 = a_1s \\
= \; & \Big((\delta a)s_1 - a_1(\delta s) - (\delta a_1)s + a(\delta s_1)\Big)ss_1 \\
= \; & \Big((\delta(as_1)) - (\delta(a_1s))\Big)ss_1 \\
= \; & \Big(\delta(as_1 - a_1s)\Big)ss_1 \\
= \; & \Big(\delta(0)\Big)ss_1 \\
= \; & 0.
\end{aligned}$$

Thus $\delta$ is well-defined. We now prove that $\delta$ is a derivation. First we prove that $\delta$ is a linear map. It is clear that $\delta$ preserves scalar multiplication, so we need only check that it preserves addition. Suppose that $as^{-1}, a_1s_1^{-1} \in R$, where $s \neq 0 \neq s_1$. Then

$$\begin{aligned}
& \delta(as^{-1} + a_1s_1^{-1}) \\
= \; & \delta\Big((as_1 + a_1s)(ss_1)^{-1}\Big) \\
= \; & \Big(\delta\Big(as_1 + a_1s\Big)ss_1 - (as_1 + a_1s)(\delta(ss_1))\Big)s^{-2}s_1^{-2} \\
= \; & \Big(\Big(\delta(s_1)a + s_1(\delta a) + (\delta s)a_1 + s(\delta a_1)\Big)ss_1 \\
& -(as_1 + a_1s)\Big((\delta s)s_1 + s(\delta s_1)\Big)\Big)s^{-2}s_1^{-2} \\
= \; & \Big(\delta(s_1)ass_1 + \delta(a)ss_1^2 + \delta(s)a_1ss_1 + \delta(a_1)s^2s_1 - a\delta(s)s_1^2 - a_1ss_1\delta(s) \\
& -a\delta(s_1)ss_1 - \delta(s_1)a_1s^2\Big)s^{-2}s_1^{-2} \\
= \; & \Big(\delta(a)ss_1^2 - a\delta(s)s_1^2 + \delta(a_1)s^2s_1 - a_1\delta(s_1)s^2\Big)s^{-2}s_1^{-2} \\
= \; & \Big(\delta(a)s - a\delta(s)\Big)s^{-2} + \Big(\delta(a_1)s_1 - a_1\delta(s_1)\Big)s_1^{-2} \\
= \; & \delta(as^{-1}) + \delta(a_1s_1^{-1}), \quad \text{as required.}
\end{aligned}$$

Therefore, $\delta$ is a linear map. We next prove that $\delta$ satisfies the multiplicative condition for derivations.

$$\delta\Big(as^{-1}.a_1s_1^{-1}\Big) \;=\; \delta\Big(aa_1(ss_1^{-1})\Big)$$

$$
\begin{aligned}
&= \Big(\delta(aa_1)ss_1 - aa_1\delta(ss_1)\Big)s^{-2}s_1^{-2} \\
&= \Big(\delta(a)a_1ss_1 + a\delta(a_1)ss_1 - aa_1\delta(s)s_1 - aa_1s\delta(s_1)\Big)s^{-2}s_1^{-2} \\
&= \Big(a_1s_1\Big(\delta(a)s - a\delta(s)\Big) + as\Big(\delta(a_1)s_1 + a_1\delta(s_1)\Big)\Big)s^{-2}s_1^{-2} \\
&= (a_1s_1^{-1})\Big(\delta(a)s - a\delta(s)\Big)s^{-2} \\
&\quad +(as^{-1})\Big(\delta(a_1)s_1 - a_1\delta(s_1)\Big)s_1^{-2} \\
&= (a_1s_1^{-1})\delta(as^{-1}) + (as^{-1})\delta(a_1s_1^{-2}), \quad \text{as required.}
\end{aligned}
$$

Therefore, $\delta$ is a derivation of $R$.

This calculation shows that, in particular, $D_\varphi^S$ may be extended to a derivation $D_\varphi$ of $R$. Clearly, $D_\varphi$ still extends $\varphi$. The sets $V$ and $F$ together form a generating set for $R$, as a field. However, we have defined $D_\varphi$ on this generating set (since all derivations of $R$ map $F$ to $0$), and we have extended $D_\varphi$ from $V$ and $F$ to $R$ in a unique manner. Therefore, $D_\varphi$ is unique. Since the definition which we made for $\delta(rs^{-1})$ is the only one possible, the uniqueness of $D_\varphi$ is clear.

Also, since $D_\varphi$ is an extension of a derivation of $S$, it maps $S$ into itself. This completes the proof of the theorem. ■

**Corollary 3.4.4** *Let $X$ and $Y$ be endomorphisms of $V$ and $D_X, D_Y$ be the derivations of $R$ which extend them (as in Theorem 3.4.3). Then, the derivation which extends the endomorphism $[X,Y] := XY - YX$ is $[D_X, D_Y] := D_XD_Y - D_YD_X$.*

**Proof** : We know, by lemma 2.2.3 that $[D_X, D_Y]$ is a derivation of $R$. Moreover, it clearly extends $[X,Y]$. Thus, by the uniqueness of the derivation from Theorem 3.4.3, this must be the unique derivation extending $[X,Y]$. Thus, the corollary is proved. ■

## 3.5 Derivations of R

We now apply the notions of section 3.4 to the dual space of $V$, $V^*$. In section 3.4 we extended linear maps from $V$ to $S$ to derivations of the rational expressions in elements of $V$. With the identifications we made before, when we consider these notions in $V^*$ instead of $V$, we will be able to extend any linear map from $V^*$ to $S^*$ to a derivation of $\mathbf{R}$, the field of rational functions over $V$ (since we identified $\mathbf{R}$ with $R^*$).

**Definition 3.5.1** *Let $V$ be a vector space over $F$ and let $X$ be a linear transformation of $V$. Let $V^*$ be the dual of $V$. We define ${}^tX : V^* \mapsto V^*$ to be the linear transformation of $V^*$ defined as follows*

$$({}^tX(f))(a) = f(Xa)$$

*for all $f \in V^*$ and all $a \in V$. ${}^tX$ is called the* transpose of $X$.

*The derivation of $\mathbf{R}$ which extends $-{}^tX$ (as in Theorem 3.4.3) is called the* derivation canonically associated with $X$.

The following theorem is important.

**Theorem 3.5.2** *Let $V$ be a vector space over $F$, and $V^*$ the dual of $V$. Let $X$ and $Y$ be linear transformations of $V$, denote by $\Delta_X$ and $\Delta_Y$ the derivations of $\mathbf{R}$ canonically associated with $X$ and $Y$ respectively (as in Definition 3.5.1). Then, the mapping $X \mapsto \Delta_X$ of $\mathfrak{E}$ into the space of derivations of $\mathbf{R}$ is linear, and $\Delta_{[X,Y]} = [\Delta_X, \Delta_Y]$ (where $\Delta_{[X,Y]}$ is the derivation canonically associated with $[X,Y]$. Thus, the mapping $X \mapsto \Delta(X)$ is a Lie homomorphism.*

**Proof** : The first assertion is clear, since all the steps which contribute to the construction of $\Delta_X$ from $X$ preserve linearity.

To prove the second assertion, we investigate the properties of the transpose map. If $f \in V^*$ and $a \in V$,

$$\begin{aligned}
\Big( {}^t(XY)(f)\Big)(a) &= f(XYa) \\
&= \Big( {}^tX(f)\Big)(Ya) \\
&= \Big(({}^tY)({}^tX)(f)\Big)(a),
\end{aligned}$$

so ${}^t(XY) ={}^t Y {}^tX$. Then,

$$\begin{aligned}
{}^t[X,Y] &= {}^t(XY) -{}^t (YX) \\
&= {}^tY{}^tX -{}^t X{}^tY \\
&= -[{}^tX,{}^tY].
\end{aligned}$$

Therefore, $-{}^t[X,Y] = [{}^tX,{}^tY] = [-{}^tX, -{}^tY]$. Thus, the derivation $\Delta_{[X,Y]}$, extending $-{}^t[X,Y] = [-{}^tX, -{}^tY]$ is the same as $[\Delta_X, \Delta_Y]$, by Corollary 3.4.4. This completes the proof of the theorem. ■

**Definition 3.5.3** *Let* $\mathbf{R}$ *be the field of rational functions over* $V$*, and suppose that* $x \in V$*. The linear function* $\lambda \mapsto \lambda(x)$ *from* $V^*$ *into* $F$ *may be considered as a linear function of* $V^*$ *into* $S^*$ *(since* $F$ *is embedded canonically in* $S^*$*). This linear function extends to a derivation,* $D_x$ *of* $\mathbf{R}$*. We will call* $D_x$ *the* partial derivation of $\mathbf{R}$ with respect to $x$.

*If* $H \in \mathbf{R}$ *and* $H$ *is defined at a point* $y \in V$*, then the function* $x \mapsto (D_xH)(y)$ *from* $V$ *into* $F$ *is called the* differential of $H$ at the point $y$. *If* $E$ *is the set of points of* $V$ *at which* $H$ *is defined, we define the function* $dH : E \times V \to F$ *by*

$$(dH)(y, x) := (D_xH)(y)$$

**Theorem 3.5.4** *Let* $V$ *be a vector space over* $F$ *and let* $\mathbf{R}$ *be the algebra of rational functions over* $V$*. If* $X$ *is a linear transformation of* $V$*, denote by* $D_X$ *the derivation of* $\mathbf{R}$ *which canonically extends* $X$*. If* $H \in \mathbf{R}$ *and* $H$ *is defined at* $y \in V$*, then* $D_XH$ *is defined at* $y$ *and we have*

$$(D_XH)(y) = -(dH)(y, Xy) \tag{3.2}$$

**Proof** : We first prove that $D_XH$ is defined at $y$. We can write $H = FG^{-1}$ where $F$ and $G$ are polynomial functions on $V$ and where $G(y) \neq 0$. Well, we then have that

$$D_XH = \frac{(D_XF)G - F(D_XG)}{G^2},$$

which is defined at $y$ precisely because $G(y) \neq 0$.

Suppose $H$ is a linear function. Since $D_x$ is an extension of $\lambda \to \lambda(x)$, we have that $(dH)(y, x) = (D_xH)(y) = H(x)$. Thus, if $H$ is a linear function,

$$\begin{aligned}(D_XH)(y) &= (-{}^tX(H))(y) \\ &= -H(Xy) \\ &= -(dH)(y, Xy), \quad \text{as required.}\end{aligned}$$

Therefore, we have 3.2 in the case of linear functions. If $H$ is a constant function, then both sides of 3.2 are zero. Our result now follows from the fact that both sides of 3.2 act in the same way under products and quotients. That is to say, if $H = FG$, then

$$\begin{aligned}(D_XH) &= (D_XF)G + F(D_XG) \quad \text{and,} \\ (dH)(y, Xy) &= ((dF)(y, Xy))(G(y)) + F(y)((dG)(y, Xy)).\end{aligned}$$

Also, if $H = FG^{-1}$, then

$$\begin{aligned} (D_X H)(y) &= \frac{((D_X F)(y)(G(y)) - F(y)((D_X G)(y))}{(G(y))^2} \quad \text{and,} \\ (dH)(y, Xy) &= \frac{((dF)(y, Xy))G(y) - F(y)((dG)(y, Xy))}{(G(y))^2} \end{aligned}$$

Since they also act the same way under the vector space operations, and $\mathbf{R}$ is generated by the linear functions and the constants by the vector space operations and products and quotients, we have 3.2 for all $H \in \mathbf{R}$. This completes the proof of the Theorem. ■

**Lemma 3.5.5** *Let $H$ be a rational function on $V$, and $T$ be a linear transformation of $V$. Suppose that $H$ is defined at $T(y)$ for some $y \in V$. Then, for all $x \in V$,*

$$\Big(d(H \circ T)\Big)(y, x) = (dH)\Big(T(y), T(x)\Big) \tag{3.3}$$

**Proof** : Both $H$ and $H \circ T$ are rational functions on $V$. If $H$ is a constant, then both sides of 3.3 are zero. Suppose then that $H$ is a linear function. Then, the right-hand side of 3.3 is equal to $H(T(x))$. Let $\{v_i\}$ be a basis for $V$. Then $T = \sum T_i v_i$, where the $T_i : V \to F$ are linear. Then $H \circ T = \sum T_i H(v_i)$, whence

$$\begin{aligned} \Big(d(H \circ T)\Big)(y, x) &= \sum (dT_i)(y, x) H(v_i) \\ &= \sum T_i(x) H(v_i) \\ &= H(T(x)), \end{aligned}$$

so that 3.3 holds in this case. We now show that addition products and quotients all preserve the identity 3.3. From this it will follow that 3.3 holds for all rational functions on $V$, since $\mathbf{R}$ is generated, as a field, by linear functions. So, suppose that $H_1$ and $H_2$ are rational functions on $V$ for which 3.3 holds. Then, we show that 3.3 holds for $(H_1 + H_2) \circ T$, $(H_1 H_2) \circ T$ and, if $H_1(T(y)) \neq 0$, for $((H_1^{-1}) \circ T)$ (assuming here that $H_1$ and $H_2$ are defined at $T(y)$).

So,

$$\begin{aligned} \Big(d((H_1 + H_2) \circ T)\Big)(y, x) &= \Big(d(H_1 \circ T)\Big)(y, x) + \Big(d(H_2 \circ T)\Big)(y, x) \\ &= (dH_1)\Big(T(y), T(x)\Big) + (dH_2)\Big(T(y), T(x)\Big) \\ &= \Big(d(H_1 + H_2)\Big)\Big(T(y), T(x)\Big), \quad \text{as required.} \end{aligned}$$

Secondly,

$$\begin{aligned}
&\Big(d((H_1H_2)\circ T)\Big)(y,x)\\
&= \Big(d\Big((H_1\circ T)(H_2\circ T)\Big)\Big)(y,x)\\
&= \Big(\Big(d(H_1\circ T)\Big)(y,x)\Big)\Big(H_2\circ T\Big)(y,x)\\
&\quad +\Big(H_1\circ T\Big)(y,x)\Big(\Big(d(H_1\circ T)\Big)(y,x)\Big)\\
&= \Big((dH_1)\Big(T(y),T(x)\Big)\Big)H_2(T(y)) + H_1(T(y))\Big((dH_2)\Big(T(y),T(x)\Big)\Big)\\
&= \Big(d(H_1H_2)\Big)\Big(T(y),T(x)\Big) \quad \text{as required.}
\end{aligned}$$

Finally,

$$\begin{aligned}
\Big(d\Big((H_1^{-1})\circ T\Big)\Big)(y,x) &= \Big(d\Big((H_1\circ T)^{-1}\Big)\Big)(y,x)\\
&= \frac{-\Big(d(H_1\circ T)\Big)(y,x)}{\Big((H_1\circ T)(y)\Big)^2}\\
&= \frac{-(dH_1)\Big(T(y),T(x)\Big)}{(H_1(T(y)))^2}\\
&= -(d(H_1^{-1}))\Big(T(y),T(x)\Big).
\end{aligned}$$

Therefore, 3.3 is preserved under the operations of $\mathbf{R}$ , and so holds for all rational function $H$ on $V$. This completes the proof of the Lemma. ■

## 3.6 Algebraic groups

For this section, as before, we denote our ground field by $F$, a finite dimensional vector space over $F$ by $V$ and the space of linear transformations of $V$ by $\mathfrak{E}$ . We denote the algebra of polynomial functions over $\mathfrak{E}$ by $\nu(\mathfrak{E})$, as defined in Section 3.3. We denote by $\mathfrak{R}$ the field of rational functions on $\mathfrak{E}$ .

After Chevalley [9], for the remainder of this chapter we call an element of the general linear group, $GL(V)$, over $V$ an *automorphism* of $V$. If

$V$ also possesses the structure of an algebra, then an automorphism which also preserves the multiplicative structure of $V$ will be called an *algebra automorphism.*

**Definition 3.6.1** *Let $G \leq GL(V)$. If there exists a set $A \subseteq \nu(\mathfrak{E})$ such that*

$$G = \{s \in GL(V) \mid P(s) = 0 \;\forall P \in A\},$$

*then we call $G$ an* algebraic group. *We call the set $A$ a* defining set of $G$.

**Definition 3.6.2** *If $E$ is a subset of $\mathfrak{E}$, then the ideal of $\nu(\mathfrak{E})$ generated by all elements of $\nu(\mathfrak{E})$ which are zero on $E$ is called the* ideal associated with $E$.

It is easy to see that if $\mathfrak{a}$ is the ideal associated with $E$ then

$$\mathfrak{a} = \{P \in \nu(\mathfrak{E}) \mid P(s) = 0 \;\forall s \in E\}.$$

**Example 3.6.3** *If $A$ has the structure of an algebra over $F$, then the multiplicative group of all algebra automorphisms of $A$ forms an algebraic group.*

Let $G$ be the group of all algebra automorphisms of $A$. Then, an automorphism of $A$ is contained in $G$ if and only if

$$u\Big((sx)(sy) - s(xy)\Big) = 0$$

for all $u \in \mathfrak{E}$. However, if $x$, $y$ and $u$ are fixed, the function

$$s \mapsto u\Big((sx)(sy) - s(xy)\Big)$$

is easily seen to be in $\nu(\mathfrak{E})$. Thus, $G$ is an algebraic group.

**Example 3.6.4** *Let $V_1$ and $V_2$ be subspaces of $V$ such that $V_2 \subseteq V_1$. The automorphisms $s$ of $V$ such that $sx \equiv x \pmod{V_2}$ for all $x \in V_1$ form an algebraic group.*

Let $G$ be the set of all such automorphisms. It is obvious that $G$ is a multiplicative group. Now, if $\lambda$ is a linear function on $V$ which is zero on $V_2$, and if $x \in V_1$, then the function

$$s \mapsto \lambda(sx)$$

is a linear function on $\mathfrak{E}$ . We will denote this function by $u_{\lambda,x}$. Now, the elements of $G$ are those $s$ such that $sx - x \in V_2$ for all $x \in V_1$. This is the set of all $s$ such that $\lambda(sx - x) = 0$ for all linear functions $\lambda$ on $V$ which are zero on $V_2$. Thus, the elements of $G$ are the automorphisms of $V$ such that

$$u_{\lambda,x}(s) - \lambda(x) = 0$$

for all the functions $u_{\lambda,x}$. Thus, $G$ is the set of zeroes for a set of polynomial functions on $V$, and so is an algebraic group.

## 3.7 Generalised and Generic points

Suppose that $L$ is an extension field of $F$. Then we define $V^L$ to be the tensor product $L \otimes_F V$, which we identify with the vector space spanned by elements of $V$ with coefficients in $L$. $V$ is canonically embedded in $V^L$, so we write $V \subseteq V^L$. We write $\mathfrak{E}^L$ for the algebra of linear transformations of $V^L$. A system of co-ordinate functions for $V$ is also a system of co-ordinate functions for $V^L$. Thus, since $\mathfrak{E}$ is spanned by the system of co-ordinate functions, $\mathfrak{E}^L$ is the space of linear transformations of $V^L$ and we again write $\mathfrak{E} \leq \mathfrak{E}^L$. This means that any linear transformation of $V$ can be uniquely extended to a linear transformation of $V^L$.

We can consider $\nu(\mathfrak{E})$ to be the polynomials over a system of co-ordinate functions for $V$ with coefficients in $F$. Thus, it is clear that, considering $\nu(\mathfrak{E}^L)$ to be polynomials in the same system of co-ordinate functions with coefficients in $L$, that $\nu(\mathfrak{E}^L)$ can be canonically identified with $\nu(\mathfrak{E})^L$. We make this identification. Finally, we write $\mathfrak{R}^L$ for the field of rational functions over $\mathfrak{E}^L$, this identification being similar.

Since each linear transformation of $V$ may be uniquely extended to a linear transformation of $V^L$, if $G$ is an algebraic group of automorphisms of $V$, then $G$ is a group of automorphisms of $V^L$. However, $G$ is not, in general, an algebraic group with respect to $V^L$. Therefore, we make

**Definition 3.7.1** *Let $V$ be a vector space over $F$, $L$ an extension field of $F$ and $G$ an algebraic group of automorphisms of $V$. We define $G^L$ to be the smallest algebraic group of automorphisms of $V^L$ which contains $G$.*

If $V_1$ and $V_2$ are subspaces of $V$ such that $V = V_1 \oplus V_2$, then we can associate the algebra of polynomials on $V$ with $\nu(\mathfrak{E}_1) \otimes \nu(\mathfrak{E}_2)$, where $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are the sets of endomorphisms on $V_1$ and $V_2$, respectively (see [9], p81). The function which realises this isomorphism is $\varphi : P \otimes P' \mapsto \Big((s, s') \mapsto P(s)P'(s')\Big)$. This is a standard property of the tensor product. We make this identification in the following lemma.

**Lemma 3.7.2** *Let $V'$ be a vector space over $F$, $\mathfrak{E}'$ the space of endomorphisms of $V'$, and $\nu(\mathfrak{E}')$ and $\nu(\mathfrak{E} \times \mathfrak{E}')$ the algebras of polynomial functions on $\mathfrak{E}'$ and $\mathfrak{E} \times \mathfrak{E}'$ respectively. Suppose $E \subseteq \mathfrak{E}$ and $E' \subseteq \mathfrak{E}'$, and suppose that $\mathfrak{a}$ and $\mathfrak{a}'$ are the ideals associated with $E$ and $E'$ respectively. Then, the ideal associated with $E \times E'$ in $\nu(\mathfrak{E} \times \mathfrak{E}')$ is*

$$\mathfrak{m} = \nu(\mathfrak{E}) \otimes \mathfrak{a}' + \mathfrak{a} \otimes \nu(\mathfrak{E}').$$

*If $E$ and $E'$ are the sets of all points which are zero on $\mathfrak{a}$ and $\mathfrak{a}'$ respectively, then the set of all points zero on $\mathfrak{m}$ is $E \times E'$.*

**Proof** : Let $B$ be a basis for $\nu(\mathfrak{E})$ which contains a basis, $A$, for $\mathfrak{a}$ . Let $C$ be the set $B \backslash A$. If $M \in \nu(\mathfrak{E} \times \mathfrak{E}')$, then

$$M = \sum_{i=1}^{n} P_i \otimes U_i + \sum_{j=1}^{m} Q_j \otimes V_j$$

where $P_1, \dots P_n$ are distinct elements of $A$, $Q_1 \dots Q_m$ are distinct elements of $C$ and $U_1, \dots U_n, V_1 \dots, V_m$ are elements of $\nu(\mathfrak{E}')$. If $(s, s') \in \mathfrak{E} \times \mathfrak{E}'$, then

$$M(s, s') = \sum_{i=1}^{n} P_i(s)U_i(s') + \sum_{j=1}^{m} Q_j(s)V_j(s').$$

Suppose that $M$ is zero on $E \times E'$. If $s'_0 \in E'$, then for all $s \in E$, $M(s, s'_0) = 0$, $P_i(s) = 0$ for $1 \leq i \leq n$, whence

$$\sum_{j=1}^{m} Q_j(s)U_j(s'_0) = 0.$$

Therefore, $\sum_{j=1}^{m} U_j(s'_0)Q_j \in \mathfrak{a}$. But, the $Q_j$ are linearly independent with respect to $\mathfrak{a}$ , so $U_j(s'_0) = 0$ for all $1 \leq j \leq m$. Since this is true for all $s'_0 \in E'$, the $U_j$ are contained in $\mathfrak{a}'$, so $M \in \mathfrak{m}$. It is clear that any element of $\mathfrak{m}$ is zero on $E \times E'$.

Now suppose that $E$ is the set of all points of $\mathfrak{E}$ which $\mathfrak{a}$ vanishes on, and likewise for $E'$ and $\mathfrak{a}'$. Let $(s,s')$ be a point of $\mathfrak{E}\times\mathfrak{E}'$ which vanishes on $\mathfrak{m}$ . If $P\in\mathfrak{a}$, then $P\otimes 1\in\mathfrak{m}$ and

$$0=\Big(P\otimes 1\Big)(s,s')=P(s)$$

so $s\in E$. Similarly, if $P'\in\mathfrak{a}'$ then $1\otimes P'\in\mathfrak{m}$ and

$$0=\Big(1\otimes P'\Big)(s,s')=P'(s')$$

so $s'\in E'$. This completes the proof of the Lemma. ■

If $s\in\mathfrak{E}$, then we define the function $\eta(s):\nu(\mathfrak{E})\to\nu(\mathfrak{E})$ by

$$\eta(s)P(t)=P(st)$$

for all $P\in\nu(\mathfrak{E})$, and all $t\in\mathfrak{E}$. $\eta(s)$ is an endomorphism of $\nu(\mathfrak{E})$.

**Lemma 3.7.3** *If $\mathfrak{a}$ is a vector subspace of $\nu(\mathfrak{E})$, and $s$ is an invertible element of $\mathfrak{E}$ such that $\eta(s)$ maps $\mathfrak{a}$ into itself, then $\eta(s)$ maps $\mathfrak{a}$* onto *itself.*

**Proof** : If $s,t\in\mathfrak{E}$, then $\eta(st)=\eta(s)\circ\eta(t)$, and if $s$ is the identity automorphism of $V$ then $\eta(s)$ is the identity mapping on $\nu(\mathfrak{E})$. Thus, if $s$ is invertible,

$$\eta(s^{-1})=\Big(\eta(s)\Big)^{-1}$$

and $\eta(s)$ is invertible.

The algebra $\nu(\mathfrak{E})$ is isomorphic to the symmetric algebra of the dual $\mathfrak{E}^*$ of $\mathfrak{E}$ . This algebra has a natural grading by degree. It is clear that if $P$ is a linear map on $\mathfrak{E}$, then so is $\eta(s)P$. Therefore, $\eta(s)$ maps $\mathfrak{E}^*$ into itself. For $n\geq 1$, denote by $\nu_n$ the set of elements of $\nu(\mathfrak{E})$ which are homogeneous of degree $n$ and by $\nu'_n$ the set $\sum_{m\leq n}\nu_m$. Then it is clear that $\eta(s)$ maps $\nu_n$ into itself for all $n$. Thus $\eta(s)$ maps $\mathfrak{a}\cap\nu'_n$ into itself for all $n$. Since $\eta(s)$ is invertible, its kernel is trivial. The spaces $\nu'_n$ are finite-dimensional, as $V$ is finite-dimensional. Thus, $\eta(s)$ maps $\mathfrak{a}\cap\nu'_n$ onto itself for all $n$. The Lemma follows from the fact that $\mathfrak{a}$ is the union of the sets $\mathfrak{a}\cap\nu'_n$ for $0\leq n<\infty$. ■

**Lemma 3.7.4** *Let $G_0 \subseteq GL(V)$ be closed under the product in $\mathfrak{E}$ . If $\mathfrak{a}$ is the ideal associated with $G_0$ and $P \in \mathfrak{a}$, then*

$$P(st) = \sum_{i=1}^{n} P_i(s)A_i(t) + \sum_{j=1}^{n} A'_j(s)P_j(t) \tag{3.4}$$

*for all $s, t \in \mathfrak{E}$, where $A_1, \ldots, A_n, A'_1, \ldots, A'_n \in \mathfrak{E}$ and $P_1, \ldots, P_n \in \mathfrak{a}$. If $G$ is the set of all automorphisms of $V$ for which $P(s) = 0$ for all $P \in \mathfrak{a}$ then $G$ is an algebraic group whose associated ideal is $\mathfrak{a}$ . If $s \in GL(V)$ then $s \in G$ if and only if $\eta(s)$ maps $\mathfrak{a}$ into itself.*

**Proof** : $G_0$ is closed under multiplication, so the function $(s, t) \mapsto P(st)$ of $\mathfrak{E} \times \mathfrak{E}$ into $F$ is certainly zero on $G_0 \times G_0$. Then 3.4 follows immediately from Lemma 3.7.2. If $s \in G$ then $P_i(s) = 0$ for $1 \leq i \leq n$ so $\eta(s)$ maps $\mathfrak{a}$ into itself.

Now since $s$ is invertible, by Lemma 3.7.3 $\eta(s)$ maps $\mathfrak{a}$ onto itself. Therefore, if $P \in \mathfrak{a}$, there exists some $P' \in \mathfrak{a}$ such that

$$P'(st) = \eta(s)P'(t) = P(t), \quad \text{for all } t \in \mathfrak{E}.$$

Then, $P(I) = P'(s) = 0$, since $s \in G$. Therefore, $I \in G$.

Suppose $s \in GL(V)$ is such that $\eta(s)$ maps $\mathfrak{a}$ into itself. Then, for all $P \in \mathfrak{a}$,

$$P(s) = \eta(s)P(I) = 0,$$

so $s \in G$. It remains to show that $G$ is an algebraic group with associated ideal $\mathfrak{a}$ .

It is clear that the set of automorphisms which map $\mathfrak{a}$ into itself form a group. This set is $G$, so $G$ is a group. However, by definition $G$ is the set of automorphisms which vanish on $\mathfrak{a}$ , so $G$ is an algebraic group. This completes the proof of the Lemma. ■

We now prove some properties of $G^L$.

**Lemma 3.7.5** *Let $V$, $L$, $G$ and $G^L$ be as in Definition 3.7.1. If $\mathfrak{a}$ is the ideal associated with $G$, then the ideal associated with $G^L$ is $\mathfrak{a}^L$. Also, $G = G^L \cap \mathfrak{E}$.*

**Proof** : If $P \in \mathfrak{a}^L$, then $P(x) = 0$ for all $x \in G$. Suppose that $P \in \nu(\mathfrak{E}^L)$ and that $P(x) = 0$ for all $x \in G$. Let $\{l_i\}_{i \in I}$ be a basis for $L$ with respect to $F$. Then,

$$P = \sum_{i \in I} l_i P_i$$

for some $P_i \in \nu(\mathfrak{E})$. We do not suppose that $I$ is a finite set, but the above sum must have finite support (all but a finite number of the $P_i$ are zero).

Since $P(x) = 0$ for all $x \in G$, and the $l_i$ are linearly independent over $F$, $P_i(x) = 0$ for all $x \in G$ and all $i \in I$. Thus, $P_i \in \mathfrak{a}$ for all $i \in I$, whence $P \in \mathfrak{a}^L$. Therefore, if $P \in \nu(\mathfrak{E}^L)$, $P(x) = 0$ for all $x \in G$ if and only if $P \in \mathfrak{a}^L$.

It is clear that $\mathfrak{a}^L$ is an ideal of $\nu(\mathfrak{E})^L$, since $\nu(\mathfrak{E})^L$ is spanned by $\nu(\mathfrak{E})$ as an $L$-space. Let $G_0$ be the algebraic group of automorphisms of $V^L$ defined by the set $\mathfrak{a}^L$ ($G_0$ exists by Lemma 3.7.4). Clearly, the ideal associated with $G_0$ is $\mathfrak{a}^L$.

It is clear that $G \leq G_0$. Suppose that $G_1$ is an algebraic group of automorphisms of $V^L$ such that $G \leq G_1$. If $\mathfrak{b}$ is the ideal associated with $G_1$, $\mathfrak{b} \leq \mathfrak{a}^L$, since $\mathfrak{b}$ is zero on $G$. Therefore, $G_0 \leq G_1$. Thus, $G_0$ is an algebraic group of automorphisms of $V^L$ which is contained in every other algebraic group of automorphisms of $V^L$ containing $G$, meaning $G_0 = G_L$. Hence, the ideal associated with $G^L$ is $\mathfrak{a}^L$. Obviously, $G \subseteq G_L \cap \mathfrak{E}$. So, suppose that $x \in G_L \cap \mathfrak{E}$. Then, $P(x) = 0$ for all $P \in \mathfrak{a}^L$, which means that $P(x) = 0$ for all $P \in \mathfrak{a}$. Therefore, $x$ is an automorphism of $\mathfrak{E}$ such that $P(x) = 0$ for all $P \in \mathfrak{a}$. Thus, $x \in G$, by the definition of $\mathfrak{a}$ , so $G^L \cap \mathfrak{E} \subseteq G$, completing the proof of the lemma. ∎

**Definition 3.7.6** *Let $G$ be an algebraic group of automorphisms of $V$, and $L$ an extension field of $F$. If $s \in G^L$, then we call $s$ a* generalised point *of $G$.*

*Suppose $s$ is a generalised point of $G$ such that if $P \in \nu(\mathfrak{E})$ and $P(s) = 0$ then $P(x) = 0$ for all $x \in G$. Then we call $s$ a* generic point *of $G$.*

If $P \in \nu(\mathfrak{E})$ is such that $P(x) = 0$ for all $x \in G$, then $P \in \mathfrak{a}$, so that $P \in \mathfrak{a}^L$ and $P(s) = 0$ for all $s \in G^L$. A generic point is one for which this implication also goes the other way.

**Definition 3.7.7** *An algebraic group of automorphisms of $G$ is called* irreducible *if the ideal associated with $G$ is prime (see [12], p167).*

**Theorem 3.7.8** *Let $G$ be an algebraic group of automorphisms of $V$. There exists a unique algebraic subgroup $G_1 \leq G$ which is irreducible and of finite index in $G$.*

**Proof** : Let $\nu(\mathfrak{E})$ be the algebra of polynomials on $\mathfrak{E}$ , and let $\mathfrak{a}$ be the ideal of $\nu(\mathfrak{E})$ associated with $G$. Denote by $\mathfrak{p}$ the set of polynomials, $P$, for which there exists a polynomial $Q$ with $Q(I) \neq 0$ such that $PQ$ is zero on $G$. It is clear that $\mathfrak{a} \leq \mathfrak{p}$. We show that $\mathfrak{p}$ is an ideal of $\nu(\mathfrak{E})$.

Let $P, P' \in \mathfrak{p}$. Then, there are $Q, Q'$, with $Q(I) \neq 0$ and $Q'(I) \neq 0$ such that $PQ, P'Q' \in \mathfrak{a}$. But, $QQ'(I) \neq 0$, and $(P-P')(QQ') \in \mathfrak{a}$, so that $P - P' \in \mathfrak{p}$. The product of any element of $\nu(\mathfrak{E})$ by any element of $\mathfrak{p}$ is an element of $\mathfrak{p}$ , so $\mathfrak{p}$ is an ideal of $\nu(\mathfrak{E})$.

Since $\nu(\mathfrak{E})$ is isomorphic to a polynomial algebra in a finite number of variables over $F$, all ideals of $\nu(\mathfrak{E})$ have a finite number of generators. Let $\{P_1, \ldots, P_n\}$ be a generating set for $\mathfrak{p}$ , and let $Q_i$ be such that $Q_i(I) \neq 0$ and $P_iQ_i \in \mathfrak{a}$ for $1 \leq i \leq n$. Define $Q_0 = Q_1 \ldots Q_n$. Then $Q_0(I) \neq 0$, and since all elements of $\mathfrak{p}$ can be written in the form $A_1P_1 + \ldots A_nP_n$ for $A_1, \ldots, A_n \in \nu(\mathfrak{E})$, we have $PQ_0 \in \mathfrak{a}$ for all $P \in \mathfrak{p}$.

Let $G_1$ be the set of automorphisms, $s$, of $V$ such that $P(s) = 0$ for all $P \in \mathfrak{p}$. Since $\mathfrak{a} \leq \mathfrak{p}$, $G_1 \subseteq G$. We show that $G_1$ is a group. Suppose $s \in G_1$ and $P \in \mathfrak{p}$. Then $PQ_0 \in \mathfrak{a}$. By Lemma 3.7.4, $\eta(s)(PQ) = (\eta(s)P)(\eta(s)Q) \in \mathfrak{a}$. If $Q_0(s) \neq 0$ then $\eta(s)Q_0$ is not zero at $I$, so $\eta(s)P \in \mathfrak{p}$. Then $Q_0(s)\eta(s)P \in \mathfrak{p}$. If $Q_0(s) = 0$, then $Q_0(s)\eta(s)P$ is the zero function, and so clearly in $\mathfrak{p}$ . In any case, $Q_0(s)\eta(s)P \in \mathfrak{p}$. Thus, this function is zero on $G_1$, by definition of $G_1$.

If $t \in G_1$, then the function $s \mapsto Q_0(s)P(st)$ is in $\mathfrak{a}$ . Since $Q_0(I) \neq 0$, we see that the function $s \mapsto P(st)$ is in $\mathfrak{p}$ , and thus zero on $G_1$. Thus, $G_1$ is closed under multiplication. Therefore, by Lemma 3.7.4, $G_1$ is an algebraic group.

All $P \in \mathfrak{p}$ are zero on $G_1$. Conversely, suppose that $P$ is zero on $G_1$. Let $s \in G$. If $Q_0(s) \neq 0$, then $P(s) = 0$ for all $P' \in \mathfrak{p}$ (since $Q_0P'$ is zero on $G$ for all $P' \in \mathfrak{p}$), so $s \in G_1$. Therefore, for all $s \in G$, $Q_0(s)P(s) = 0$. Hence $Q_0P \in \mathfrak{a}$ and so $P \in \mathfrak{p}$. Thus, the ideal associated with $G_1$ is $\mathfrak{p}$ .

We now show that $\mathfrak{p}$ is a prime ideal, so that $G_1$ is irreducible. Suppose that $P_1$ and $P_2$ are such that $P_1P_2 \in \mathfrak{p}$, $P_2 \notin \mathfrak{p}$. Then, there exists $s \in G_1$ such that $P_2(s) \neq 0$. Since $P_1P_2 \in \mathfrak{p}$, the function $(\eta(s)P_1)(\eta(s)P_2)$ is in $\mathfrak{p}$ , thus $Q_0(\eta(s)P_1)(\eta(s)P_2) \in \mathfrak{a}$. But, $Q_0.(\eta(s)P_2)$ is not zero at $I$, so $\eta(s)P_1 \in \mathfrak{p}$. But this proves that $P_1 = (\eta(s^{-1}))(\eta(s)P_1) \in \mathfrak{p}$. Thus, $\mathfrak{p}$ is a prime ideal and $G_1$ is irreducible.

It remains to show that $G_1$ is of finite index in $G$ and that it is the unique irreducible subgroup of $G$ of finite index. Let $\mathfrak{q}$ be the ideal of $\nu(\mathfrak{E})$ generated by the functions $\eta(t^{-1})Q_0$ where $t \in G$. The ideal $\mathfrak{q}$ is finitely generated, so let $\{M_1, \ldots, M_k\}$ generate $\mathfrak{q}$ . For $1 \leq i \leq k$, we write

$M_i = \sum_{j=1}^{k'} A_{ij}\Big(\eta(t_j^{-1})Q_0\Big)$, where $A_{ij} \in \nu(\mathfrak{E})$ and $t_1 \ldots, t_{k'} \in G$. Since $Q_0(I) \neq 0$, $\eta(t^{-1})Q_0 \neq 0$ for all $t \in G$. Thus, for all $t \in G$, there exists some $M_i$ which is not zero at $t$, and so there is some $j$ such that $Q_0(t_j^{-1}t) \neq 0$. But, all points of $G$ for which $Q_0 \neq 0$ are contained in $G_1$. Thus, for all $t \in G$, there is some $1 \leq j \leq k'$ such that $t_j^{-1}t \in G_1$. This is equivalent to saying that the cosets $t_jG_1$, $1 \leq j \leq k'$ cover $G$. Thus, $G_1$ is of finite index in $G$.

Let $G_1'$ be an irreducible algebraic subgroup of finite index in $G$, and let $\mathfrak{p}'$ be the ideal associated with $G_1'$. It is obvious that $\mathfrak{a} \leq \mathfrak{p}'$, but that $Q_0 \notin \mathfrak{p}'$, since $Q_0(I) \neq 0$. If $P \in \mathfrak{p}$, then $PQ_0 \in \mathfrak{a} \leq \mathfrak{p}'$. But $Q_0 \notin \mathfrak{p}'$ so $P \in \mathfrak{p}'$ since $\mathfrak{p}'$ is prime. Thus, $\mathfrak{p} \subseteq \mathfrak{p}'$.

Let $G_1', t_1'G_1', \ldots, t_m'G_1'$ be the distinct cosets of $G_1'$ in $G$. For $1 \leq i \leq m$, $t_i' \notin G_1'$, so there exists a $Q_i' \in \mathfrak{p}'$ such that $Q_i'(t_i'^{-1}) \neq 0$. Define $Q'$ to be the product of the functions $\eta(t_j'^{-1})Q_j'$ for $1 \leq j \leq m$. Then $Q'(I) \neq 0$ but $Q'$ is zero on each of the sets $t_j'G_1'$ $1 \leq j \leq m$. Suppose that $P'$ is an element of $\mathfrak{p}'$. $P'$ is zero on $G_1$, so $P'Q'$ is zero on $G$. But then, by the definition of $\mathfrak{p}$ , $P' \in \mathfrak{p}$, so that $\mathfrak{p}' \subseteq \mathfrak{p}$. Therefore, $\mathfrak{p}' = \mathfrak{p}$ and $G_1' = G_1$. This completes the proof of the Theorem. ∎

**Definition 3.7.9** *The group $G_1$ from Theorem 3.7.8 is called the* algebraic component of the identity element of $G$.

## 3.8 Polynomial and Rational functions on $G$

Let $V$ be a vector space and $\mathfrak{E}$ its space of endomorphisms. Let $\nu(\mathfrak{E})$ be the ring of polynomials over $\mathfrak{E}$ .

**Definition 3.8.1** *Let $G$ be an algebraic group of automorphisms of $V$. Let $\mathfrak{a}$ be the ideal of $\nu(\mathfrak{E})$ associated with $G$. The set of functions from $G$ to $F$ which are the restrictions to $G$ of polynomial functions from $\nu(\mathfrak{E})$ to $F$ are called* polynomial functions on $G$. *We denote the ring of polynomials on $G$ by $\nu(G)$.*

It is clear that the ring $\nu(G)$ is isomorphic to $\nu(\mathfrak{E})/\mathfrak{a}$. Suppose that $G$ is an irreducible algebraic group. Then $\mathfrak{a}$ is a prime ideal which is equivalent to saying that $\nu(\mathfrak{E})/\mathfrak{a}$ is an integral domain (see [12],p167). Therefore, if $G$ is an irreducible algebraic group, we can form the field of quotients of $\nu(G)$.

Let $\mathfrak{R}_G$ be the field of quotients of $\nu(G)$. Then, as we did in the case of $\nu(\mathfrak{E})$, we define rational functions on $G$ as follows.

Let $R \in \mathfrak{R}_G$. If $s \in G$, we say that $R$ is *defined at* $s$ if we can write $R = PQ^{-1}$, where $P, Q \in \nu(G)$ and $Q(s) \neq 0$. The value $P(s)(Q(s))^{-1}$ does not depend on the choice of $P$ and $Q$. We call this value the *value of* $R$ *at* $s$ and denote it by $R(s)$.

If $E$ is a non-empty subset of $G$, we say that a function $H : E \to F$ is a *rational function on* $G$ if there is an $R \in \mathfrak{R}_G$ such that $E$ is the set of points of $G$ at which $R$ is defined and if, for all $s \in E$ we have $H(s) = R(s)$.

If $L$ is an extension field of $F$ then we define the rational functions over $G^L$ in the obvious way. All rational functions over $G^L$ are $L$-linear combinations of rational functions over $G$. That is to say, if $\mathfrak{R}_G$ is the space of rational functions over $G$, then $\mathfrak{R}_G^L$ is the space of rational functions over $G^L$.

We use this notion in the proof of the following Theorem.

**Theorem 3.8.2** *For an algebraic group of automorphisms of* $V$ *to admit a generic point, it is necessary and sufficient for the group to be irreducible.*

**Proof** : Let $\mathfrak{a}$ be the ideal associated with $G$. Suppose that $s$ is a generic point for $G$. If $P$ is a polynomial, then $P(s) = 0$ if and only if $P \in \mathfrak{a}$. So, suppose that $P_1, P_2 \notin \mathfrak{a}$. Then $P_1(s) \neq 0$ and $P_2(s) \neq 0$, whence $(P_1P_2)(s) \neq 0$, so that $P_1P_2 \notin \mathfrak{a}$. Thus, $\mathfrak{a}$ is a prime ideal and $G$ is irreducible.

Conversely, suppose that $G$ is irreducible. Let $L$ be the field of rational functions on $G$ (which we can construct since $G$ is irreducible). Let $\{u_{ij}\}_{1 \leq i,j \leq n}$ be a system of co-ordinate functions for $\mathfrak{E}$ , and let $u_{ij}^*$ be the function induced by $u_{ij}$ on $G$. It is clear that $u_{ij}^* \in L$ for $1 \leq i, j \leq n$.

We also consider the $u_{ij}$ as forming a system of co-ordinate functions for $\mathfrak{E}^L$. Therefore, by the definition of co-ordinate functions, there is some $s \in \mathfrak{E}^L$ such that

$$u_{ij}(s) = u_{ij}^*$$

for $1 \leq i, j \leq n$. If $P \in \nu(\mathfrak{E})$, then the function $P(s)$ is the function induced by $P$ on $G$. Therefore, $P(s) = 0$ if and only if $P$ is zero on $G$. If we prove that $s \in G^L$ then $s$ will obviously be a generic point of $G$.

If $t \in \mathfrak{E}^L$, denote by $D(t)$ the determinant of $t$. It is clear that $D$ is not zero on $G$ (and that $D$ is a polynomial function on $\mathfrak{E}^L$), proving that $D(s) \neq 0$. Thus $s$ is an automorphism of $V^L$. The ideal associated with

$G^L$, $\mathfrak{a}^L$, is composed of $L$-linear combinations of elements of $\mathfrak{a}$, so that $\mathfrak{a}^L$ is zero at $s$. Thus, by the definition of $\mathfrak{a}^L$, $s \in G^L$. Therefore $G$ has a generic point, namely $s$. This completes the proof of the Theorem. ■

## 3.9 The Lie algebra of an algebraic group

In this section, we show that to each algebraic group of automorphism of $V$ we can identify a Lie algebra which is a subset of the Lie algebra of endomorphisms over $V$. When $F = \mathbb{R}$ or $F = \mathbb{C}$ and algebraic groups are Lie groups, then the Lie algebra of $G$ as an algebraic group is the same as the Lie algebra of $G$ as a Lie group.

We saw in section 3.5 that any linear map from $V^*$ to $V^*$ could be extended to a derivation of $R^*$ (which we identify with the field of rational functions on $V$). In this section, $\mathfrak{E}$ will take the role of $V$. Thus, we know that any linear map from $\mathfrak{E}^*$ to itself may be extended to a derivation of $\mathfrak{R}$, the field of rational functions on $\mathfrak{E}$. This extension will be crucial in constructing the Lie algebra associated with an algebraic group.

Since $\mathfrak{E}$ is an associative algebra, we can turn it into a Lie algebra by defining the Lie product

$$[X, Y] = XY - YX.$$

We call the Lie algebra so obtained $\mathfrak{gl}(V)$.

Suppose $X \in \mathfrak{E}$. We associate $X$ with an endomorphism $f_X : \mathfrak{E} \to \mathfrak{E}$, by defining $f_X(s) = Xs$ for all $s \in \mathfrak{E}$. Motivated by this, we make the following definition:

**Definition 3.9.1** *If $X \in \mathfrak{E}$, let $\delta(X)$ be the derivation of $\mathfrak{R}$ which is canonically associated with the endomorphism $f_X : s \mapsto Xs$ of $\mathfrak{E}$ (see Definition 3.5.1).*

We can now define the Lie algebra of an algebraic group.

**Definition 3.9.2** *Let $G$ be an algebraic group of automorphisms of $V$ and let $\mathfrak{a}$ be the ideal of $\nu(\mathfrak{E})$ associated with $G$. The set of endomorphisms, $X$, of $V$ for which $\delta(X)$ maps $\mathfrak{a}$ into itself is called the* Lie algebra of $G$, *and we denote it by $\mathfrak{g}$.*

This definition is quite abstract and we spend the rest of this section trying to give a more explicit characterisation of $\mathfrak{g}$ .

We must show that the name *Lie algebra* is appropriate for $\mathfrak{g}$ , that is to say $\mathfrak{g}$ is actually a Lie algebra.

By tracing through all the constructions involved in making $\delta(X)$, it is clear that $\delta(X+Y) = \delta(X) + \delta(Y)$, and that if $\alpha \in F$, $\delta(\alpha X) = \alpha\delta(X)$ (defining the scalar multiplication for functions in the obvious way). Thus, $\mathfrak{g}$ is a subspace of $\mathfrak{gl}(V)$ . It remains to show that $\mathfrak{g}$ is closed under the Lie product. So we prove that, if $X$ and $Y$ are in $\mathfrak{g}$ , so is $[X,Y]$. That is to say, that if $\delta(X)$ and $\delta(Y)$ map $\mathfrak{a}$ into itself, then so does $\delta([X,Y])$. We show that $\delta([X,Y]) = [\delta(X), \delta(Y)]$. If this is true, then $\delta([X,Y])$ clearly maps $\mathfrak{a}$ into itself.

Firstly, for all $s, X, Y \in \mathfrak{E}$,

$$\begin{aligned}
[f_X, f_Y]s &= f_X(f_Y s) - f_Y(f_X s) \\
&= f_X Y s - f_Y X s \\
&= XYs - YXs \\
&= [X,Y]s \\
&= f_{[X,Y]}s,
\end{aligned}$$

so that the function $X \mapsto f_X$ preserves Lie products. We calculate ${}^t f_{[X,Y]}$. If $g \in \mathfrak{E}^*$ and $s \in \mathfrak{E}$, then

$$\begin{aligned}
{}^t f_{[X,Y]}(g)(s) &= g(f_{[X,Y]}s) \\
&= g([f_X, f_Y]s)
\end{aligned}$$

Thus, by Theorem 3.5.2, the mapping $X \mapsto \delta(X)$ preserves the Lie product. Thus, we have proved the following theorem.

**Theorem 3.9.3** *If $G$ is an algebraic group, then $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$ .* ∎

We prove some properties of $\delta$ which allows us to give a more explicit description of the elements of $\mathfrak{g}$ .

**Lemma 3.9.4** *Let $R$ be a rational function on $\mathfrak{E}$ , which is defined at a point $s \in \mathfrak{E}$. Let $X \in \mathfrak{E}$. Then $\delta(X)R$ is defined at $s$ and*

$$\begin{aligned}
(\delta(X)R)(s) &= -(dR)(s, Xs) \quad \textit{and,} \\
(\delta(X)u)(s) &= -u(Xs), \quad \textit{if } u \textit{ is a linear function on } \mathfrak{E}.
\end{aligned}$$

**Proof** : The lemma follows immediately from Theorem 3.5.4 and from the fact that, if $u$ is a linear function on $\mathfrak{E}$ , $(du)(s, Xs) = u(Xs)$ for all $s \in \mathfrak{E}$. ∎

**Definition 3.9.5** *Let $t \in GL(V)$. Define the automorphism $\eta(t)$ of $\mathfrak{R}$ by setting $\eta(t)R$ to be the composition of the mapping $s \mapsto ts$ from $\mathfrak{E}$ into itself with $R$. Similarly, we define $\eta'(t)$ by composing the mapping $s \mapsto st$ with $R \in \mathfrak{R}$.*

The definition of $\eta(t)$ as an automorphism of rational functions is obviously an extension of the definition of $\eta(t)$ which we made for polynomial functions.

**Lemma 3.9.6** *If $X \in \mathfrak{E}$ and $t$ is an automorphism of $V$ then*

$$(\eta(t))^{-1}\delta(X)\eta(t) = \delta(tXt^{-1}) \tag{3.5}$$

$$(\eta'(t))^{-1}\delta(X)\eta'(t) = \delta(X). \tag{3.6}$$

**Proof** : Suppose that $u$ is a linear function on $\mathfrak{E}$ , and that $s \in \mathfrak{E}$. Then

$$\begin{aligned}
\Big((\eta(t))^{-1}\delta(X)\eta(t)\Big)(u)(s) &= (\eta(t))^{-1}\delta(X)(\eta(t)u)(s) \\
&= (\eta(t))^{-1}\delta(X)(u(ts)) \\
&= (\eta(t))^{-1}(\delta(X)u(ts)) \\
&= -(\eta(t))^{-1}(u(Xts)), \quad \text{by Lemma 3.9.4,} \\
&= -u(t^{-1}Xts) \\
&= \delta(t^{-1}Xt)(u)(s).
\end{aligned}$$

Thus, the two sides of 3.5 agree on a set of generators for $\mathfrak{R}$ . But, if $u, v \in \mathfrak{R}$ and $s \in \mathfrak{E}$, then

$$\begin{aligned}
&\Big((\eta(t))^{-1}\delta(X)\eta(t)\Big)(uv)(s) \\
&= \Big((\eta(t))^{-1}\delta(X)\Big)(uv)(ts) \\
&= (\eta(t))^{-1}\Big((\delta(X)u(ts))v(ts) + u(ts)(\delta(X)v(ts))\Big) \\
&= \Big((\eta(t))^{-1}\delta(X)u(ts)\Big)v(t^{-1}ts) + u(t^{-1}ts\Big((\eta(t))^{-1}\delta(X)v(ts)\Big) \\
&= \Big((\eta(t))^{-1}\delta(X)\eta(t)\Big)(u)(s)v(s) + u(s)\Big((\eta(t))^{-1}\delta(X)\eta(t)\Big)(v)(s),
\end{aligned}$$

so that

$$(\eta(t))^{-1}\delta(X)\eta(t)(uv) = \Big((\eta(t))^{-1}\delta(X)\eta(t)u\Big)v + u\Big(\eta(t))^{-1}\delta(X)\eta(t)v\Big).$$

Thus $(\eta(t))^{-1}\delta(X)\eta(t)$ is a derivation. Since $\delta(t^{-1}Xt)$ is also a derivation, and these two derivations agree on a generating set of $\mathfrak{R}$, they must be equal. Therefore 3.5 is proved.

The proof of 3.6 is similar. ■

If $L$ is an extension field of $F$, the function $\delta(X)R^L : \mathfrak{R}^L \to \mathfrak{R}^L$ is the function which extends $\delta(X)R$. It is straightforward to check that the formulas of Lemma 3.9.4 are true if $s \in \mathfrak{E}^L$ and if $X \in \mathfrak{E}^L$ and that Lemma 3.9.6 still holds if we let $X \in \mathfrak{E}^L$, $t \in GL(V^L)$ and $R \in \mathfrak{R}^L$.

We now prove a theorem which gives us a more explicit characterisation of the Lie algebra of an algebraic group.

**Theorem 3.9.7** *Let $G$ be an algebraic group of automorphisms of $V$ and let $\mathfrak{a}$ be the ideal associated with $G$. Let $s$ be a generalised point of $G$ and $\mathfrak{g}$ the Lie algebra of $G$. Suppose that $X \in \mathfrak{E}$. Then $X \in \mathfrak{g}$ if and only if either one of the following two conditions holds;*

$$(dP)(s, Xs) = 0 \quad \textit{for all } P \in \mathfrak{a}; \tag{3.7}$$

$$(dP)(s, sX) = 0 \quad \textit{for all } P \in \mathfrak{a}. \tag{3.8}$$

**Proof** : Suppose that $X \in \mathfrak{g}$ and that $P \in \mathfrak{a}$. Then

$$(\delta(X)P)(s) = -(dP)(s, Xs)$$

by Lemma 3.9.4 (and its extension to extension fields of $F$). But, since $\delta(X)$ maps $\mathfrak{a}$ into itself, $(\delta(X)P)(s) = 0$. Thus, $(dP)(s, Xs) = 0$.

Conversely, suppose that $(dP)(s, Xs) = 0$ for all $P \in \mathfrak{a}$. Let $L$ be an extension field of $F$ such that $s \in G^L$. Suppose that $t \in G^L$. By 3.6

$$\eta'(t)\delta(X)P = \delta(X)\eta'(t)P.$$

If $P \in \mathfrak{a}$, then $\eta'(t)P \in \mathfrak{a}^L$, the ideal associated with $G^L$ (see Lemma 3.7.5). Therefore,

$$(\delta(X)\eta'(t)P)(s) = 0$$

which implies that

$$(\eta'(t)\delta(X)P)(s) = 0.$$

But, this is equivalent to $(\delta(X)P)(st) = 0$. If $x \in G$, then by setting $t = s^{-1}x \in G^L$,

$$(\delta(X)P)(x) = 0,$$

for all $x \in G$, so $\delta(X)P \in \mathfrak{a}$. Thus, $X \in \mathfrak{g}$. Thus, 3.7 is a necessary and sufficient condition for $X \in \mathfrak{g}$.

We now prove that condition 3.7 is equivalent to condition 3.8. Since $\eta(s)P$ is the composition of the endomorphism $t \mapsto st$ of $\mathfrak{E}^L$ and of $P$, we apply Lemma 3.5.5 to yield

$$(d\eta(s)P)(I, X) = (dP)(s, sX),$$

where $I$ denotes the identity automorphism (the identity in $G$). If $x \in G$ then $sx \in G^L$ so $\eta(s)P(x) = P(sx) = 0$, for all $P \in \mathfrak{a}^L$. Thus, $\eta(s)$ maps $\mathfrak{a}^L$ into itself. Therefore, for 3.8 to be satisfied, it is necessary and sufficient that $(dP)(I, X) = 0$ for all $P \in \mathfrak{a}^L$. But this is equivalent to 3.7, since $I$ is trivially a generalised point of $G$. This completes the proof of the theorem. ■

**Corollary 3.9.8** *Let $G$ be an algebraic group of automorphisms of $V$, and let $H$ be an algebraic subgroup of $G$. Then the Lie algebra of $H$ is contained in the Lie algebra of $G$.*

**Proof** : Let $I$ be the identity element of $G$. If $X$ is an element of the Lie algebra of $H$, we have $(dP)(I, X) = 0$ for all polynomial functions $P$ on $V$ which are in the ideal associated with $H$ (by Theorem 3.9.7). But, since $H \leq G$, it is clear that the ideal associated with $G$ is contained in the ideal associated with $H$. Thus, we have $(dP)(I, X) = 0$ for all $P$ in the ideal associated with $G$. Therefore, $X$ is contained in the Lie algebra of $G$. This completes the proof of the Corollary. ■

## 3.10 Derivations of $V^L$

Suppose that $L$ is an extension field of $F$. Suppose that $D$ is a derivation of $L$ (where $L$ is considered as a vector space over $F$, not necessarily finite-dimensional).

With $D$, associate a function $D : V^L \to V^L$ in the following way: if $x \in V^L$ define $Dx$ to be the point of $V^L$ such that $u(Dx) = D(u(x))$ for all $u \in V^*$. This makes sense because any linear function from $V$ to $F$ can be canonically extended to a linear function from $V^L$ to $L$. If we consider $V^L$ to be the space $L \otimes_F V$, then the mapping $D : V^L \to V^L$ is actually the mapping $D \otimes \iota$, where $\iota$ is the identity map on $V$. This is easy to check.

$Dx = 0$, for all $x \in V$, since the original derivation mapped $F$ to $0$.

**Lemma 3.10.1** *With the notation as above, suppose that $V$ has the structure of an algebra over $F$. Then $V^L$ possesses the structure of an algebra over $L$. If $x, y \in V^L$ then*

$$D(xy) = (Dx)y + x(Dy),$$

*so $D$ is a derivation of $V^L$.*

**Proof** : Write $x = \sum_i a_i x_i$, $y = \sum_j b_j y_j$ where $a_i, b_j \in L$ and $x_i, y_j \in V$. Then,

$$xy = \sum_i \sum_j a_i b_j x_i y_j,$$

whence,

$$\begin{aligned} D(xy) &= \sum_i \sum_j (Da_i) b_j x_i y_j + \sum_i \sum_j a_i (Db_j) x_i y_j \\ &= (Dx)y + x(Dy), \quad \text{as required.} \end{aligned}$$

Thus the lemma is proved. ■

**Lemma 3.10.2** *With the same notation as in Lemma 3.10.1, also let $S$ be a rational function on $V$, which we identify with the rational function on $V^L$ which extends $S$. If $S$ is defined at $x \in V^L$, then*

$$D(S(x)) = (dS)(x, Dx). \tag{3.9}$$

**Proof** : The rational functions on $V^L$ which are defined at $x$ form a subring $\mathbf{R}_x$ of the field of rational functions on $V^L$. The function $S \mapsto S(x)$ is then a homomorphism from $\mathbf{R}_x$ into $L$. If $S$ is a linear function on $V$, then

$$\begin{aligned} D(S(x)) &= S(D(x)) \quad \text{and,} \\ (dS)(x, Dx) &= S(D(x)) \end{aligned}$$

so that 3.9 holds in this case. Suppose that $S = S_1 S_2$ and that 3.9 holds for $S_1$ and $S_2$. Then

$$\begin{aligned} D(S(x)) &= D((S_1S_2)(x)) \\ &= D(S_1(x)S_2(x)) \\ &= D(S_1(x))S_2(x) + S_1(x)D(S_2(x)) \\ &= (dS_1)(x, Dx)S_2(x) + S_1(x)(dS_2)(x, Dx) \\ &= (d(S_1S_2))(x, Dx) \\ &= (dS)(x, Dx). \end{aligned}$$

Thus, 3.9 holds for all polynomials. Now suppose that $S$ is a rational function defined at $x$, so that there are polynomials $P$ and $Q$ such that $SQ = P$ and $Q(x) \neq 0$. Then

$$\begin{aligned} D(S(x))Q(x) + S(x)D(Q(x)) &= D(P(x)) \quad \text{and,} \\ ((dS)(x, Dx))Q(x) + S(x)((dQ)(x, Dx)) &= (dP)(x, Dx). \end{aligned}$$

It follows immediately that $D(S(x)) = (dS)(x, Dx)$, since $Q(x) \neq 0$. Thus, 3.9 holds for all rational functions $S$, and the lemma is proved. ■

**Theorem 3.10.3** *Let $G$ be an algebraic group of automorphisms of $V$, and $L$ be an extension field of $F$. Let $s \in G^L$, and $\mathfrak{g}$ be the Lie algebra of $G$. If $D$ is a derivation of $L$, then $(Ds)s^{-1} \in \mathfrak{g}^L$, the Lie algebra of $G^L$.*

**Proof** : Let $\mathfrak{a}$ be the ideal associated with $G$, and suppose $P \in \mathfrak{a}$. By Lemma 3.10.2,

$$\begin{aligned} (dP)(s, Ds) &= D(P(s)) \\ &= D(0) \\ &= 0. \end{aligned}$$

By Lemma 3.9.7, on replacing $X$ by $(Ds)s^{-1}$,

$$\begin{aligned} (dP)(s, (Ds)s^{-1}s) &= (dP)(s, Ds) \\ &= 0, \end{aligned}$$

for all $P \in \mathfrak{a}$. If $P \in \mathfrak{a}^L$, then, by the linearity of $(dP)$, $(dP)(s, Ds) = 0$. Thus, $(Ds)s^{-1} \in \mathfrak{g}^L$. This completes the proof of the theorem. ■

**Theorem 3.10.4** *Suppose that $G$ is an irreducible algebraic group of automorphisms of $V$ and that $\mathfrak{g}$ is the Lie algebra of $G$. Suppose that $s$ is a generic point of $G$. Then, for all $X \in \mathfrak{g}^{F(s)}$ there is a unique derivation $D_X$ of $F(s)$ such that $D_X s = Xs$ ($F(s)$ is defined in Definition 3.2.1).*

**Proof**: Let $\mathfrak{R}_G$ be the field of rational functions on $G$. We prove that the function $\varphi : R \mapsto R(s)$ is an isomorphism from $\mathfrak{R}_G$ to $F(s)$. We first show that $\varphi$ is defined. That is to say that if $R$ is a rational function over $G$, then $R$ is defined at $s$. But if $R$ were not defined at $s$, then any representation of $R$ as $PQ^{-1}$ with $P, Q \in \nu(G)$ would have $Q(s) = 0$. Then $Q(x) = 0$ for all $x \in G$, since $s$ is a generic point of $G$. Thus, $R$ is not defined at any point of $G$, and so is not a function on $G$. Therefore, all rational functions on $G$ are defined at $s$, and $\varphi$ is well-defined.

Then, $\varphi$ is obviously a homomorphism. Suppose $R \in \mathfrak{R}$ is such that $R(s) = 0$. Then, if $R = PQ^{-1}$, where $P, Q \in \nu(G)$ then $P(s) = 0$. Since $s$ is a generic point of $G$, this implies that $P(x) = 0$ for all $x \in G$. Therefore, $P \in \mathfrak{a}$, the ideal associated with $G$, and $R$ is the zero function on $G$. Thus, the kernel of $\varphi$ is zero, and $\varphi$ is a monomorphism.

We now prove that $\varphi$ is an epimorphism. Suppose that $x \in F(s)$. Then $x = yz^{-1}$ where $y, z \in F[s]$. Since $F[s]$ is generated as a ring by $F$ and by the elements $u_{ij}(s)$, we can consider $x$ and $z$ as polynomials in the $u_{ij}(s)$ with coefficients in $F$. But polynomials over $\mathfrak{E}$ , and so polynomials over $G$, can be considered as polynomials in the functions $u_{ij}$. If $w$ is a polynomial in the $u_{ij}(s)$ with coefficients in $F$, write $w^*$ for the polynomial over $\mathfrak{E}$ obtained by replacing $u_{ij}(s)$ by $u_{ij}$. Then, $y^*(z^*)^{-1}$ maps to $x$ under $\varphi$. Therefore, $\varphi$ is an epimorphism, and so an isomorphism.

Let $X$ be an element of $\mathfrak{g}$ . Then $-\delta(X)$ is a derivation of the field of rational functions over $\mathfrak{E}$ . However, by the definition of $\mathfrak{g}$ , $\delta(X)$ (and so $-\delta(X)$) maps $\mathfrak{a}$ into itself. Therefore, $-\delta(X)$ induces a derivation of $\mathfrak{R}_G$ , and thus, by the isomorphism $\varphi$, a derivation on $F(s)$, which we denote by $D_X$. If $u$ is a linear function on $\mathfrak{E}$ ,

$$\Big(\delta(X)u\Big)(s) = -u(Xs).$$

Therefore, $D_X\Big(u(s)\Big) = u(Xs)$ for all linear functions $u$ on $\mathfrak{E}$ . Thus, $D_X s = Xs$, by the definition of $D_X s$ above.

We now prove that $D_X$ is unique. To see this, note that if $Ds = Xs$, then $D(u(s)) = u(Xs)$ for all linear functions $u$ on $\mathfrak{E}$ . In particular, $D\Big(u_{ij}(s)\Big)$ is uniquely determined. However, $F(s)$ is generated as a field by the elements $u_{ij}(s)$ and by $F$. Thus, $D$ is uniquely defined on a generating set of $F(s)$, and so on all of $F(s)$. This proves the uniqueness of $D_X$, and so completes the proof of the Theorem. ■

## 3.11 Exponentials

In Lie groups, the exponential map maps from the tangent space to the manifold, that is to say from the Lie algebra to the Lie group. We cannot define such a map explicitly, so we pass first to the ring of formal power series in a transcendental indeterminate and then we can define the exponential. When this is done, the exponential map for algebraic groups turn out to have many of the same properties as that for Lie groups.

We define the ring of formal power series, and the exponential function, $\exp(TX)$, for $X \in \mathfrak{E}$. It turns out that if $G$ is an algebraic group, then $X \in \mathfrak{g}$ if and only if $\exp(TX)$ is a generalised point of $G$. Proving this is the purpose of this section.

Let $T$ be a transcendental indeterminate which commutes with all elements of $F$. Let $F[T]$ be the ring of polynomials in $T$ with coefficients in $F$. We define a metric on $F[T]$ in the following way.

**Definition 3.11.1** *Let $F[T]$ be the ring of polynomials in $T$ (a commuting transcendental indeterminate) with coefficients in $F$. If $0 \neq x \in F[T]$, then we write*

$$x = \sum_{i=0}^{n} a_i T^i,$$

*where $a_i \in F$, and define $|x|$ to be $2^{-m}$ where $m$ is the least integer such that $a_i \neq 0$. We also define $|0| = 0$.*

The following Lemma is immediate.

**Lemma 3.11.2** *If $x, y \in F[T]$, and $0 \neq a \in F$, then*

$$\begin{aligned} |a| &= 1 \\ |x| &= 0 \quad \textit{if and only if } x = 0 \\ |x+y| &\leq \textit{max}\ \{|x|, |y|\} \\ |xy| &= |x||y|. \end{aligned}$$

**Proof** : The proof follows immediately from the definition of $|x|$. ■

The above Lemma shows that, by defining $d(x,y) = |x-y|$, for $x,y \in F[T]$, we obtain a metric on $F[T]$.

**Definition 3.11.3** *Let $\mathfrak{t}$ be the completion of $F[T]$ with respect to the above metric.*

We identify $\mathfrak{t}$ with infinite series of elements of $F[T]$ which converge, and thus with the ring of formal power series in $T$ with coefficients in $F$.

If $\{a_i\}_{1\leq i<\infty}$ is a sequence of elements of $\mathfrak{t}$ which converge to 0 in this metric, then the partial sums $\sum_{i=1}^{n} a_i$ converge for all $n$. We prove this by noting that, for all $m \geq 1$, there is some $k$ such that $|a_j| \leq 2^{-m}$ for all $j \geq k$. If $j_1, \ldots j_s$ are all greater than $k$, then $|a_{j_1} + \ldots + a_{j_s}| \leq 2^{-m}$, so that by Cauchy's Criteria (see Bourbaki, [2], p263), $\{a_i\}_{1\leq i<\infty}$ is summable (so the infinite sum exists).

Let $L$ be the field of quotients of $\mathfrak{t}$ (which is obviously an integral domain). Let $V^{\mathfrak{t}}$ be the elements of $V^L$ which can be expressed as a linear combination of elements of $V$ with coefficients in $\mathfrak{t}$. We define a metric on $V^{\mathfrak{t}}$. Suppose $x \in V^{\mathfrak{t}}$. If $u \in V^*$ then we associate $u$ with a linear function from $V^L$ into $L$ in the usual way. It is clear that $u$ maps $V^{\mathfrak{t}}$ into itself. For fixed $x$, the set $\left\{|u(x)| \;\middle|\; u \in V^*\right\}$ has a greatest element, since all elements are either zero or of the form $2^{-m}$ where $m \geq 0$. Denote this greatest element by $|x|$.

If we write $x = \sum a_i x_i$ where $a_i \in L$ and $x_i \in V$, then $u(x) = \sum a_i u(x_i)$, and so $|u(x)| \leq \max_i\{|a_i|\}$. Now suppose that the elements $x_i$ form a linearly independent set. Then, for all $i$, there exists a linear function $u_i : V \to F$ such that $u_i(x_i) = 1$ and $u_i(x_j) = 0$ if $i \neq j$. Then $u_i(x) = a_i$, so that $|x| \geq |a_i|$. Thus, if the $x_i$ are linearly independent, $|x| = \max{}_i\{|a_i|\}$.

**Lemma 3.11.4** *The function $x \mapsto |x|$ from $V^{\mathfrak{t}}$ to $F$ possesses the following properties:*

$$\begin{aligned} 0 &\leq |x| \leq 1 \\ |x| &= 0 \quad \textit{if and only if } x = 0 \\ |x| &= 1 \quad \textit{if } x \neq 0 \in V \\ |x+y| &\leq \max\{|x|,|y|\}, \quad \textit{if } x,y \in V^{\mathfrak{t}}, \textit{ and} \\ |ax| &= |a||x| \quad \textit{if } a \in \mathfrak{t} \textit{ and } x \in V^{\mathfrak{t}} \end{aligned}$$

**Proof** : The proof follows immediately from Lemma 3.11.2 and the definition of $|x|$. ■

We define $d(x,y) = |x-y|$ for $x, y \in V^{\mathfrak{t}}$ and turn $V^{\mathfrak{t}}$ into a metric space. If $\{x_1, \ldots, x_n\}$ is a basis for $V$, and $x \in V^{\mathfrak{t}}$, then $x = \sum_{i=1}^{n} a_i(x) x_i$, with $a_i(x) \in t$. Then, $|x| = \max\{|a_1(x)|, \ldots, |a_n(x)|\}$. The function $x \mapsto (a_1(x), \ldots, a_n(x))$ is a continuous bijection between $V^{\mathfrak{t}}$ and $\underbrace{\mathfrak{t} \otimes \ldots \otimes \mathfrak{t}}_{n}$. Therefore, $V^{\mathfrak{t}}$ is a complete metric space also.

Suppose that $D$ is a derivation of $L$. Then, as in section 3.10, there is a derivation, $D$, of $V^L$ such that $u(Dx) = D(u(x))$ for all $x \in V^L$ and all $u \in V^*$. If $D$ maps $\mathfrak{t}$ into itself, then its extension to $V^L$ maps $V^{\mathfrak{t}}$ into itself. There is a derivation of the ring $\mathfrak{t}$ which maps $T$ to 1. We label this derivation $D_T$. $D_T$ is given by the formula $D_T(\sum a_i T^i) = \sum i a_i T^{i-1}$. We call $D_T$ the *derivation with respect to* $T$ *in* $\mathfrak{t}$ .

Consider the case where $V$ has the structure of an algebra as well as a vector space. Then, the multiplication in $V^{\mathfrak{t}}$ is continuous with respect to this metric and $|xy| \leq |x||y|$ for all $x, y \in V^{\mathfrak{t}}$. In particular, if $V$ is an associative algebra with unit (denoted 1), and $x \in V^{\mathfrak{t}}$ then $|x^n| \leq |x|^n$. Therefore, if $|x| < 1$ (which means that the constant term of $x$ is zero), then the series $\{x^i\}_{1 \leq i < \infty}$ is summable. Letting $x^0 = 1$, as usual, define

$$\exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!}.$$

We investigate some of the properties of $\exp(x)$. Suppose $x, y \in V^{\mathfrak{t}}$ and $xy = yx$. Then,

$$\begin{aligned} \exp(x+y) &= \sum_{i=0}^{\infty} \frac{(x+y)^i}{i!} \\ &= \sum_{0 \leq i < \infty} \sum_{j+k=i} \frac{x^j y^k}{j!k!}, \quad \text{since } xy = yx, \\ &= \exp(x) + \exp(y). \end{aligned}$$

Therefore, if $|x| < 1$,

$$(\exp(x))(\exp(-x)) = 1$$

and $\exp(x)$ is invertible.

**Lemma 3.11.5** *Let $D$ be a derivation of the ring $\mathfrak{t}$ . Let $x \in V^{\mathfrak{t}}$ be such that $|x| < 1$ and $x$ commutes with $Dx$. Then,*

$$\begin{aligned} D\Big(\exp(x)\Big) &= \Big(\exp(x)\Big)Dx \\ &= (Dx)\Big(\exp(x)\Big) \end{aligned}$$

**Proof** : $D$ is a derivation of $V^{\mathfrak{t}}$ so, $D(x^k) = kx^{k-1}Dx = k(Dx)x^{k-1}$, and $D(1) = 0$, since $Dx$ and $x$ commute. Then

$$\begin{aligned} D(\exp(x)) &= D\Big(\sum_k \frac{x^k}{k!}\Big) \\ &= \sum_k \frac{D(x^k)}{k!} \\ &= \sum_k \frac{kx^{k-1}Dx}{k!} \\ &= \sum_k \frac{x^{k-1}Dx}{(k-1)!} \\ &= \exp(x)Dx \quad \text{and,} \\ D(\exp(x)) &= D\Big(\sum_k \frac{x_k}{k!}\Big) \\ &= \sum_k \frac{(Dx)x^{k-1}}{(k-1)!} \\ &= (Dx)(\exp(x)). \end{aligned}$$

Thus the lemma is proved. ∎

**Lemma 3.11.6** *If $\mathfrak{a}$ is a non-zero ideal of $\mathfrak{t}$ , then there exists an $m \geq 0$ such that $\mathfrak{a} = T^m\mathfrak{t}$.*

**Proof** : The numbers $|a|$ for $0 \neq a \in \mathfrak{a}$ are of the form $2^{-k}$ where $k \geq 0$. Let $2^{-m}$ be the maximum of these, and suppose that $a \in \mathfrak{a}$ has $|a| = 2^{-m}$.

Well, $a = bT^m(1+c)$, where $0 \neq b \in F$, and $c \in \mathfrak{t}$ has $|c| < 1$. The element $1+c$ is invertible in $\mathfrak{t}$ (its inverse is $\sum_{i=0}^{\infty}(-1)^i c^i$). Therefore, $bT^M(1+c)(1+c)^{-1} = bT^m \in \mathfrak{a}$. Then $T^m \in \mathfrak{a}$, so $T^m\mathfrak{t} \leq \mathfrak{a}$. However, for all $a' \in \mathfrak{a}$, $|a'| \leq 2^{-m}$, so that $a' = T^m d$ for some $d \in \mathfrak{t}$. Thus, $\mathfrak{a} \leq T^m\mathfrak{t}$. This completes the proof of the lemma. ∎

As usual, we specialise these results by replacing $V$ with $\mathfrak{E}$. Consider $\mathfrak{t}$ to be as before, and $\mathfrak{E}^{\mathfrak{t}}$ to be defined as for $V^{\mathfrak{t}}$. Then, we prove the following important theorem:

**Theorem 3.11.7** *Let $G$ be an algebraic group of automorphisms of $V$. Let $\mathfrak{t}$ be the ring of formal power series in $T$ with coefficients in $F$. Then, $X \in \mathfrak{E}$ is contained in $\mathfrak{g}$, the Lie algebra of $G$, if and only if* $\exp(TX)$ *is a generalised point of $G$.*

**Proof** : Suppose that $\exp(TX)$ is a generalised point of $G$. Let $L$ be the field of quotients of $\mathfrak{t}$. If $D$ is the derivation of $\mathfrak{t}$ with respect to $T$, then $D$ extends to a derivation of $L$, which we also denote by $D$. It is clear that $D(TX) = X$, so that $TX$ and $D(TX)$ commute. Thus, by Lemma 3.11.5, $D(\exp(TX)) = D(TX)\exp(TX) = X\exp(TX)$. But, since $\exp(TX)$ is a generalised point of $G$, Theorem 3.10.3 tells us that $D(\exp(TX))(\exp(TX))^{-1} \in \mathfrak{g}^L$. But

$$D(\exp(TX))(\exp(TX))^{-1} = X,$$

so $X \in \mathfrak{g}^L$. Since $X \in \mathfrak{E}$, $X \in \mathfrak{g}$.

Suppose, conversely, that $X \in \mathfrak{g}$. Let $\mathfrak{a}$ be the ideal associated with $G$. We show that $P(\exp(TX)) = 0$ for all $P \in \mathfrak{a}$, so that $\exp(TX)$ is a generalised point of $G$. Denote by $\mathfrak{a}'$ the ideal of $\mathfrak{t}$ generated by the elements $P(\exp(TX))$ for $P \in \mathfrak{a}$. By Lemma 3.10.2,

$$\begin{aligned} D(P(\exp(TX))) &= (dP)(\exp(TX), D(\exp(TX))) \\ &= (dP)(\exp(TX), X\exp(TX)). \end{aligned}$$

However, $(dP)(\exp(TX), X\exp(TX)) = -(\delta(X)P)(\exp(TX))$. Since we have $X \in \mathfrak{g}$, $\delta(X)P \in \mathfrak{g}$, so that $-(\delta(X)P)(\exp(TX)) \in \mathfrak{a}'$. Therefore, $D$ maps $\mathfrak{a}'$ into itself, since it maps the generators of $\mathfrak{a}'$ into itself. By Lemma 3.11.6, if $\mathfrak{a}' \neq 0$ then $\mathfrak{a}' = T^m\mathfrak{t}$ for some $m \geq 0$. $P$, as a polynomial on $\mathfrak{E}^L$, maps $T$ to itself. However, $P(I) = 0$, since $I \in G$, so $exp(TX) - I \in T\mathfrak{E}^t$. Therefore, $P(\exp(TX)) \in T\mathfrak{t}$. Since this is true for all $P \in \mathfrak{a}$, $\mathfrak{a}' \in T\mathfrak{t}$. Therefore $m \geq 1$. However, $D(T^m) = mT^{m-1} \in \mathfrak{a}'$ since $D$ maps $\mathfrak{a}'$ into itself. Since $T^{m-1} \notin T^m\mathfrak{t}$, we have a contradiction, so that $\mathfrak{a}' = 0$.

Thus, $P(\exp(TX)) = 0$ for all $P \in \mathfrak{a}'$, meaning that $\exp(TX)$ is a generalised point of $G$. This completes the proof of the Theorem. ■

## 3.12 Tying it all together

**Example 3.12.1** *Following from Example 3.6.4, consider the Lie algebra of the algebraic group constructed there. Recall that $V_1$ and $V_2$ were subspaces of $V$, and $V_2 \subseteq V_1$. Let $G$ be the algebraic group of all automorphisms of $V$ such that $sx \equiv x$ (mod $V_2$) for all $x \in V_1$. We calculate $\mathfrak{g}$ .*

Recall the functions $u_{\lambda,x}$ from Example 3.6.4. The ideal associated with $G$ is generated by the functions $u_{\lambda,x} - \lambda(x)$. If $X$ is an element of $\mathfrak{g}$ , then

$$\delta(X)\Big(u_{\lambda,x} - \lambda(x)\Big) = \delta(X)u_{\lambda,x}$$

since $\lambda(x)$ is associated with a constant function on $\mathfrak{C}$ . But, $\delta(X)u_{\lambda,x}$ is the function

$$s \mapsto -\lambda(Xsx).$$

This function must be contained in the ideal associated with $G$, since $X \in \mathfrak{g}$, so that it takes the value zero for $s = I$, the identity of $G$. Therefore, $\lambda(Xx) = 0$ for all $x \in V_1$ and all linear functions $\lambda$ on $V$ which are zero on $V_2$. Therefore, $X$ maps $V_1$ into $V_2$.

Suppose, conversely, that $X \in \mathfrak{C}$ maps $V_1$ into $V_2$. Let $x \in V_1$, $s \in G$ and $\lambda$ be a linear function on $V$ which maps $V_2$ to zero. Then $\lambda(Xsx) = 0$. Thus,

$$\delta(X)\Big(u_{\lambda,x} - \lambda(x)\Big) \;=\; \delta(X)u_{\lambda,x}$$

is in the ideal associated with $G$, since $\Big(\delta(X)u_{\lambda,x}\Big)(s) = -\lambda(Xsx)$ which is zero if $s \in G$. Therefore, $\delta(X)$ maps the ideal associated with $G$ into itself, so that $X \in \mathfrak{g}$. Therefore, $\mathfrak{g}$ consists of all elements of $\mathfrak{C}$ which map $V_1$ into $V_2$.

Now suppose that $V_1 = V_2$. Then $G$ becomes the group of all automorphisms which map $V_1$ into itself, and $\mathfrak{g}$ is the set of all linear transformations which map $V_1$ into itself. This ends the example.

Example 3.12.1 allows us to state the following Lemma.

**Lemma 3.12.2** *If $G$ is an algebraic group of automorphisms of $V$ which leaves a subspace $W$ of $V$ invariant, then $\mathfrak{g}$ leaves $W$ invariant also.*

**Proof** : Let $G_1$ be the group of all automorphisms of $V$ which leave $W$ invariant, and let $\mathfrak{g}_1$ be its Lie algebra. By assumption $G \leq G_1$ so, by Corollary 3.9.8, $\mathfrak{g} \leq \mathfrak{g}_1$. By Example 3.12.1, $\mathfrak{g}_1$ is the set of all linear transformations of $V$ which leave $W$ invariant. Thus, all elements of $\mathfrak{g}$ leave $W$ invariant. This completes the proof of the Lemma. ■

We now move on to proving the main theorem that we require from [9], from which Tuck's result, Theorem 3.1.1, follows.

**Theorem 3.12.3** *Let $V$ be a vector space which is also furnished with the structure of an algebra, $A$, say. The set of derivations of $A$ is then the Lie algebra of the group of (algebra) automorphisms of $A$.*

**Proof** : By Example 3.6.3, the group of algebra automorphisms of $A$ is an algebraic group of automorphisms of $A$. Let this group be labelled $G$. Let $\mathfrak{g}$ be the Lie algebra of $G$.

Suppose that $X \in \mathfrak{g}$. Let $s$ be a generic point of the algebraic component of the identity element of $G$. Define the field $L$ to be $K(s)$. Let $D$ be the derivation of $L$ such that $Ds = Xs$ (see Theorem 3.10.4).

Suppose that $x, y \in A$. Then, $(sx)(sy) = s(xy)$, since $s$ is a generic point of $G$. By Lemma 3.10.1, for all $z, z' \in A^L$, $D(zz') = (Dz)z' + z(Dz')$. Also, $(Ds)z = (Ds)z$ for all $z \in A$. For all $x, y \in A$,

$$\begin{aligned} Xs(xy) &= (Ds)(xy) \\ &= D(s(xy)) \\ &= D((sx)(sy)) \\ &= (D(sx))(sy) + (sx)(D(sy)) \\ &= ((Ds)x)(sy) + (sx)((Ds)y) \\ &= (Xsx)(sy) + (sx)(Xsy) \end{aligned}$$

Since $X$ and $s$ are linear mappings, and the multiplication in $A$ is bilinear, the above identity is also true for all $x, y \in A^L$. However, $s$ is a bijection of $A^L$ so that, for all $x', y' \in A^L$, there exist $x, y \in A^L$, such that $sx = x'$ and $sy = y'$. Therefore,

$$X(x'y') = (Xx')y' + x'(Xy')$$

so $X$ is a derivation of $A$. Thus, all elements of $\mathfrak{g}$ are derivations of $A$ .

Suppose, conversely, that $X$ is a derivation of $A$. We prove that $\exp(TX)$ preserves the multiplication of $A$. If this is the case, then $\exp(TX)$ is zero on the ideal associated with $G$, and so a generalised point of $G$. By Theorem 3.11.7, $X \in \mathfrak{g}$.

We calculate $X^k(xy)$. We prove by induction that

$$X^k(xy) = \sum_{i+j=k} \binom{k}{i} (X^i x)(X^j y) \tag{3.10}$$

where the sum is over all non-negative integers $i, j$ such that $i+j=k$, where $X^0$ is presumed to be the identity automorphism and where $\binom{k}{0} = 1$. The case for $k=0$ is trivial. Assume that it holds for $k=n$. Then,

$$\begin{aligned}
X^{n+1}xy &= X(X^n xy) \\
&= X\Big(\sum_{i+j=n} \binom{n}{i} (X^i x)(X^j y)\Big) \\
&= \sum_{i+j=n} \binom{n}{i} \Big((X^{i+1}x)(X^j y) + (X^i x)(X^{j+1} y)\Big) \\
&= \sum_{i+j=n+1} \left(\binom{n}{i-1} + \binom{n}{i}\right)(X^i x)(X^j y) \\
&= \sum_{i+j=n+1} \binom{n+1}{i} (X^i x)(X^j y), \quad \text{as required.}
\end{aligned}$$

Thus 3.10 holds for all $k \geq 1$. Now, calculate $\exp(TX)xy$.

$$\begin{aligned}
\exp(TX)(xy) &= \sum_{k=0}^{\infty} (k!)^{-1} T^k X^k(xy) \\
&= \sum_{k=0}^{\infty} (k!)^{-1} T^k \sum_{i+j=k} \binom{k}{i} (X^i x)(X^j y) \\
&= \sum_{0 \leq i,j < \infty} ((i+j)!)^{-1} \binom{i+j}{i} T^i (X^i x) T^j (X^j y) \\
&= (\exp(TX)x)(\exp(TX)y).
\end{aligned}$$

Therefore, $\exp(TX)$ preserves the multiplication of $A$. Since it is invertible, it is an automorphism of $L$ which is zero on the ideal associated with $G$, and so is a generalised point of $G$. Thus, by Theorem 3.11.7, $X \in \mathfrak{g}$. This completes the proof of the Theorem. ■

## 3.13 Applications of Chevalley

We finally prove Tuck's result.

**Theorem 3.1.1** *Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero, and suppose $G$ is the group of all (algebra) automorphisms of $L$. If $H$ is a subalgebra of $L$ which is invariant under $G$, then $H$ is a characteristic ideal of $L$.*

**Proof** : Let $G$ be the algebraic group of all (vector space) automorphisms of $L$ which leave $H$ invariant (as a subspace of $L$), and $G^\alpha$ be the group of all algebra automorphisms of $L$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{g}^\alpha$ the Lie algebra of $G^\alpha$. $G^\alpha \leq G$ (by assumption) so, by Corollary 3.9.8, $\mathfrak{g}^\alpha \leq \mathfrak{g}$. By Lemma 3.12.2, all elements of $\mathfrak{g}$ (and so all elements of $\mathfrak{g}^\alpha$) leave $H$ invariant. But, by Theorem 3.12.3, $\mathfrak{g}^\alpha$ is the algebra of all derivations of $L$. Therefore, $H$ is invariant under all derivations of $L$, completing the proof of the theorem. ■

# Chapter 4

# Wielandt length

This chapter investigates the relationship between the Wielandt length and other invariants of soluble Lie algebras. We briefly consider the Fitting length and nilpotency class as invariants. However, in relation to the Wielandt length of Lie algebras, they do not prove to be illuminating.

We then prove a result motivated by results from group theory by Bryce and Cossey, [4], which bounds the derived length in terms of the Wielandt length. The bound that we obtain is significantly lower in Lie algebras than in groups, however it is not known whether this bound is the best possible.

## 4.1 The Wielandt series

Let $F$ be a fixed field of characteristic $0$.

Given Corollary 2.4.6 and Theorem 3.1.1, we now define the Wielandt series for finite-dimensional Lie algebras over $F$.

**Definition 4.1.1** *Let $L$ be a finite-dimensional Lie algebra over $F$. Then the* Wielandt series, *$\{\omega_i(L)\}$ is defined as follows;*

$$\begin{aligned} \omega_1(L) &:= \omega(L) \quad \textit{and, for } i \geq 1, \\ \omega_{i+1}(L)/\omega_i(L) &:= \omega(L/\omega_i(L)) \end{aligned}$$

Let $L$ be a finite-dimensional Lie algebra over $F$. By Corollary 2.4.6, there is some $n \geq 1$ such that $\omega_n(L) = L$. The least such $n$ is called the *Wielandt length* of $L$.

**Definition 4.1.2** *The class $\mathfrak{W}_n$ consists of all finite-dimensional soluble Lie algebras over $F$ which have Wielandt length at most $n$.*

**Lemma 4.1.3** *If $L \in \mathfrak{W}_n$, then all subideals of $L$ are of defect at most $n$.*

**Proof** : Let $S$ be a subideal of $L$. Consider the series

$$S \leq S + \omega(L) \leq S + \omega_2(L) \leq \ldots \leq S + \omega_n(L) = L.$$

We prove that, for all $1 \leq i \leq n-1$, $S + \omega_i(L) \trianglelefteq S + \omega_{i+1}(L)$. Well,

$$(S + \omega_i(L))/\omega_i(L) \ si \ L/\omega_i(L),$$

since subideality is preserved by quotients. But all subideals of $L/\omega_i(L)$ are idealised by $\omega_{i+1}(L)/\omega_i(L)$. This is equivalent to saying that $\omega_{i+1}(L)$ idealises $S+\omega_i(L)$. Since $S$ clearly idealises $S+\omega_i(L)$, $S+\omega_i(L) \trianglelefteq S+\omega_{i+1}(L)$. So we have a subideal series starting with $S$ of length at most $n$. Thus, $S$ is of defect at most $n$. This completes the proof of the Lemma. ■

## 4.2 The subideal closure

Since the Wielandt ideal of a Lie algebra is related to the subideal structure, it will be important to consider the subideals which contain an arbitrary element. The *ideal closure* of a set $X \leq L$, which is denoted by $X^L$, is a well-known concept. It is defined to be the smallest ideal of $L$ which contains $X$. The ideal closure of $X$ is the intersection of all ideals which contain $X$. With this in mind, we define the concept of the *subideal closure* of $X$.

**Definition 4.2.1** *Suppose that $L$ is a finite-dimensional Lie algebra and that $X \subseteq L$. Then define*

$$X^{L,0} := L, \quad and,$$

*supposing that $X^{L,i}$ is defined, we define inductively,*

$$X^{L,i+1} := X^{X^{L,i}}$$

*Since $X^{L,i+1} \trianglelefteq X^{L,i}$, $X^{L,i} \, si \, L$ for all $i \geq 0$. $L$ is finite-dimensional, so series must terminate after a finite number of steps.*

*Define the* subideal closure *of $X$ in $L$, denoted by $X^{sL}$ to be the intersection of all the $X^{L,i}$ for $i \geq 0$.*

We investigate some of the properties of $X^{sL}$. The first thing to note is that, since $X^{L,i}\, si\, L$ for all $i \geq 0$ and $X^{sL} = X^{L,n}$ for some $n$, $X^{sL}\, si\, L$. The most important property of $X^{sL}$, from our point of view, is given in the following theorem.

**Theorem 4.2.2** *If $H\, si\, L$ and $X \subseteq H$, then $X^{sL} \leq H$.*

**Proof** : Suppose that $H = H_n \trianglelefteq H_{n-1} \trianglelefteq \ldots \trianglelefteq H_1 \trianglelefteq H_0 = L$. Then, we prove by induction that $X^{L,i} \leq H_i$ for $0 \leq i \leq n$. The case for $i = 0$ is trivial. Suppose that $X^{L,i} \leq H_i$. Then, $X \subseteq H_{i+1} \trianglelefteq H_i$, and $X^{L,i+1} \trianglelefteq X^{L,i} \leq H_i$. Therefore, $X \subseteq H_{i+1} \cap X^{L,i+1} \trianglelefteq X^{L,i}$. Then, by the definition of $X^{L,i+1}$, $X^{L,i+1} \leq H_{i+1} \cap X^{L,i+1}$, which is to say $X^{L,i+1} \leq H_{i+1}$. This completes the induction.

Now, $X^{sL} \leq X^{L,n} \leq H_n = H$ and the proof is complete. ■

**Corollary 4.2.3** *Let $L$ be a finite-dimensional Lie algebra over $F$. Then $y \in \omega(L)$ if and only if $[x, y] \in x^{sL}$ for all $x \in L$.*

**Proof** : Suppose that $y \in \omega(L)$. Clearly, $[x, y] \in x^{sL}$ for all $x \in L$.

Suppose, conversely, that $[x, y] \in x^{sL}$ for all $x \in L$, and let $H\, si\, L$. Then, if $x \in H$,

$$\begin{aligned} [x, y] &\leq x^{sL} \\ &\leq H, \quad \text{by Theorem 4.2.2.} \end{aligned}$$

Thus, $[H, y] \in H$ and $y \in \omega(L)$. This completes the proof of the Corollary. ■

## 4.3 The structure of $\omega(L)$; $T$-algebras

**Definition 4.3.1** *A $T$-algebra is a Lie algebra for which all subideals are ideals.*

Since $\omega(L) \trianglelefteq L$, any subideal of $\omega(L)$ is a subideal of $L$. Thus, all subideals of $\omega(L)$ are ideals, so $\omega(L)$ is a $T$-algebra. Thus we investigate the properties of soluble $T$-algebras to gain insight into the structure of $\omega(L)$.

First, the following theorem, due to Stewart [17]:

**Theorem 4.3.2** *$L$ is a soluble $T$-algebra if and only if $L$ is either abelian, or the split extension of an abelian Lie algebra by the one-dimensional algebra of all scalar multiplications.*

**Proof** : Suppose $L$ is a non-abelian soluble $T$-algebra. Then, we first prove that $L$ is metabelian.

We show that $L'$ is abelian. To see this, let $x, y \in L'$. Now, $\langle x\rangle, \langle y\rangle \, si \, L'$ since $L'$ is nilpotent. Therefore, $\langle x\rangle, \langle y\rangle \, si \, L$ and so, since $L$ is a $T$-algebra, $\langle x\rangle, \langle y\rangle \trianglelefteq L$. But then, $[x, y] \in \langle x\rangle \cap \langle y\rangle$. If $\langle x\rangle \cap \langle y\rangle \neq 0$, then it is clear that $[x, y] = 0$. In any case, $[x, y] = 0$ and $L'$ is abelian.

Now let $A$ be an ideal of $L$ maximal with respect to containing $L'$ and being abelian. Since $L$ is non-abelian and soluble, $0 \neq L' \leq A \neq L$.

Let $L = A \oplus B$ as a vector space and let $0 \neq b \in B$. If $0 \neq a \in A$, then $\langle a\rangle \trianglelefteq A \trianglelefteq L$, so $\langle a\rangle \, si \, L$ meaning that $\langle a\rangle \trianglelefteq L$. Thus,

$$[a, b] \in \langle a\rangle$$

so

$$[a, b] = \delta_a a$$

for some $\delta_a \in F$.

Suppose that $\delta_a = 0$ for all $a \in A$. Then $\langle A, b\rangle$ is abelian, contradicting the choice of $A$. Thus, there exists some $a \in A$ such that $\delta_a \neq 0$.

Let $0 \neq a' \in A$. Then,

$$[a + a', b] \in \langle a + a'\rangle,$$

but

$$[a + a', b] = \delta_a a + \delta_{a'} a',$$

which implies that $\delta_a = \delta_{a'}$. Thus, there exists $0 \neq \delta \in F$ such that

$$[a, b] = \delta a, \quad \text{for all } a \in A.$$

Suppose that $B \neq \langle b\rangle$. Let $c \in B$, $c \notin \langle b\rangle$. There exists $\gamma \in F$, $\gamma \neq 0$ such that

$$[a, c] = \gamma a$$

for all $a \in A$. Then, for all $a \in A$,

$$[a, \gamma b - \delta c] = \gamma\delta a - \delta\gamma a = 0.$$

This implies $\gamma b - \delta c = 0$, contradicting $c \notin \langle b \rangle$. Thus, $B = \langle b \rangle$. It is easy to check that $L$ takes the form of a split extension of $A$ by $B$, where $B$ is the 1-dimensional algebra of all scalar multiplications.

Thus, soluble $T$-algebras are either abelian or have the form of a split extension of an abelian Lie algebra by the 1-dimensional algebra of scalar multiplications.

It is straightforward to check that if $L$ is of either of these forms, then it is a $T$-algebra. This completes the proof of the theorem. ■

Given this result, we know everything about algebras in $\mathfrak{W}_1$. They are just those algebras described in the above theorem. We also know the structure of $\omega(L)$.

If we have a $T$-algebra which is written $A \dot{+} B$, with $A$ and $B$ as in the previous theorem, we will always consider that $B$ is generated by the element which fixes all elements of $A$. There is some such element, and using this element allows us to eliminate many constants which would otherwise obscure the methods of proof.

For future reference, we note that $F(\omega(L))$, the Fitting ideal of $\omega(L)$, is either equal to $\omega(L)$ (if $\omega(L)$ is abelian), or is an abelian ideal of co-dimension 1 in $\omega(L)$ (if $\omega(L)$ is non-abelian). Also, since $F(\omega(L))$ is a characteristic ideal of $\omega(L)$,

$$F(\omega(L)) \trianglelefteq L.$$

## 4.4 Structural Lemmas

From now on, we assume that $L$ is a finite-dimensional soluble Lie algebra over $F$.

We will need the following technical lemmas, which describe the interaction between the derived ideal and the Wielandt ideal.

**Lemma 4.4.1**

$$[F(\omega(L)), L'] = 0$$

**Proof** : Let $x \in L'$. Since $F(\omega(L))$ is abelian, if $x \in F(\omega(L))$ then $[x, F(\omega(L))] = 0$.

So suppose that $x \notin F(\omega(L))$. Since all subalgebras of $L'$ are subideals of $L'$ (since $L'$ is nilpotent), and hence subideals of $L$, by the definition of $\omega(L)$,

$$[x, \omega(L)] \subseteq \langle x \rangle$$

But

$$F(\omega(L)) \trianglelefteq L$$

and so

$$[x, F(\omega(L))] \subseteq F(\omega(L)) \cap \langle x \rangle = 0, \quad \text{since } x \notin F(\omega(L)).$$

Thus,

$$[x, F(\omega(L))] = 0 \quad \text{and so}$$
$$[L', F(\omega(L))] = 0, \quad \text{as required.}$$

This completes the proof of the Lemma. ■

**Lemma 4.4.2**

$$L' \cap \omega(L) \leq F(\omega(L))$$

**Proof** : $L'$ is nilpotent, and an ideal of $L$, therefore $L' \cap \omega(L)$ is a nilpotent ideal of $\omega(L)$. Thus,

$$L' \cap \omega(L) \leq F(\omega(L)),$$

by the definition of $F(\omega(L))$. This completes the proof of the lemma. ■

## 4.5 Wielandt length and nilpotency

Lemma 4.4.1 allows us to prove a very interesting result about nilpotent Lie algebras. In the case of nilpotent Lie algebras, the Wielandt series coincides with the ascending central series.

**Theorem 4.5.1** *If $L$ is a nilpotent Lie algebra, then $\omega(L) = Z(L)$, the centre of $L$. Thus the Wielandt series coincides with the ascending central series.*

**Proof** : We first prove $\omega(L)$ must be abelian. If $\omega(L)$ is not abelian, then by Lemma 4.3.2, there exist $b \in \omega(L)$ and $0 \neq a \in \omega(L)$ such that

$[b,a] = a$. Then, for all $n$, $[b, na] = a \neq 0$, which contradicts $L$ being nilpotent. Thus $\omega(L)$ is abelian.

Let $x \in L$ and $y \in \omega(L)$. If $x \in \omega(L)$, then $[x,y] = 0$, since $\omega(L)$ is abelian. Suppose that $x \notin \omega(L)$. Then, because $L$ is nilpotent, $\langle x \rangle\, si\, L$. Therefore, by the definition of $\omega(L)$, $[x,y] \in \langle x \rangle$. But $\omega(L) \trianglelefteq L$ so

$$[x,y] \in \langle x \rangle \cap \omega(L) = 0.$$

Thus $[\omega(L), L] = 0$. Since $\zeta(L) \leq \omega(L)$, the Theorem is proved. ■

The analogous result is not true for groups, and this result allows us to leave the study of the Wielandt structure of nilpotent Lie algebras here. The Wielandt series of a nilpotent Lie algebra is equal to the ascending central series, and so the Wielandt series offers no new information.

Another invariant of Lie algebras is the Fitting length. Denote by $\mathfrak{N}$ the class of all finite-dimensional nilpotent Lie algebras, and define the product and powers of classes as in [1], pp18-19, then the Fitting length of a soluble Lie algebra $L$ is the least $n$ such that $L \in \mathfrak{N}^n$.

For the case of groups (with Fitting length defined in an exactly analogous way), Bryce and Cossey [4] proved that if a group $G$ has Wielandt length $n$, then it has Fitting length at most $n+1$.

In the case of Lie algebras, things are both better and worse. There is a better bound on the Fitting length. However, this bound is two. All finite-dimensional soluble Lie algebras $F$ have Fitting length at most 2. In fact, because of Theorem 2.1.12, they are nilpotent-by-abelian. Thus, for Lie algebras, the Fitting length also turns out to be uninteresting.

## 4.6 Wielandt length and Derived length

We now prove our result about the derived length in terms of the Wielandt length.

**Lemma 4.6.1** *If $L \in \mathfrak{W}_n$ then $L'$ is nilpotent of class at most $n$, for all $n \geq 1$.*

**Proof** : We prove the result by induction on $n$. For $n = 1$, the result is trivial, since then $L$ is a $T$-algebra, hence metabelian, and therefore $L'$ is abelian (so of nilpotence class at most 1).

Suppose that, for $k \geq 2$, $A \in \mathfrak{W}_k$ implies that $A'$ is nilpotent of class at most $k$, and let $L \in \mathfrak{W}_{k+1}$.

By the definition of the Wielandt series, $L/\omega(L) \in \mathfrak{W}_k$. Therefore, by induction $(L/\omega(L))'$ is nilpotent of length at most $k$. But, this is precisely the same as saying that $(L')^{k+1} \in \omega(L)$.

So,

$$\begin{aligned} (L')^{k+2} &= [(L')^{k+1}, L'] \\ &\leq [L' \cap \omega(L), L'] \\ &\leq [F(\omega(L)), L'] \quad \text{by Lemma 4.4.2} \\ &= 0 \quad \text{by Lemma 4.4.1} \end{aligned}$$

Thus, $L'$ is nilpotent of class at most $k+1$, and the proof of the lemma is complete. ∎

**Theorem 4.6.2** *If $L \in \mathfrak{W}_n$, then $L$ is soluble of derived length at most $d+1$ where $d$ is the least integer greater than or equal to $log_2(n+1)$.*

**Proof** : By Theorem 2.1.7, if a Lie algebra is nilpotent of class $n$, then it is soluble of class at most $d$, where $d$ is the least integer greater than or equal to $log_2(n+1)$. By Lemma 4.6.1, if $L \in \mathfrak{W}_n$, then $L'$ is soluble of derived length at most $d$, where $d$ is as in the statement of the theorem. Thus the theorem is proved, since the derived length of $L$ is one greater than the derived length of $L'$. ∎

It is not possible that we can dispense with the case of derived length as we did that of Fitting length (ie with a constant bound), since there are finite-dimensional soluble Lie algebras over $F$ with arbitrarily large derived length. The bound given in Theorem 2.1.7 on derived length in terms of nilpotency class is sharp (consider the relatively free nilpotent Lie algebra over $F$ of nilpotency class $n$ on 2 generators). Therefore, given Theorem 4.5.1, the bounding function of derived length in terms of Wielandt length, $n$, must be at least $log_2(n+1)$. Thus, for each $n$, there are only two possibilities for the best possible bound. It is not known what the best possible bound is, but it seems at least plausible that soluble Lie algebras can have longer derived length than nilpotent Lie algebras, in terms of Wielandt length. Thus, it is quite likely that the bound given in Theorem 4.6.2 is the best possible bound.

# Chapter 5

# Soluble Lie algebras of Wielandt length 2

Once again, let $F$ be a fixed field of characteristic 0.

In this chapter we characterise the finite-dimensional soluble Lie algebras over $F$. Our method of characterisation is to give a basis for $L$ as a vector-space, and then define the Lie product between basis vectors, assuming a linear extension to arbitrary products. Since $[x, x] = 0$ and $[x, y] = -[y, x]$ for all $x, y \in L$, we assume that these are always true, and do not explicitly mention this. By Jacobson [13], p4, once we have defined these products, all we need to do to check that we have a Lie algebra is to check that the Jacobi identity holds for basis elements.

## 5.1 Characterising $\mathfrak{W}_2$

Due to Theorem 4.3.2, we have a complete characterisation of the soluble Lie algebras of Wielandt length 1. In this vein, we consider Lie algebras of Wielandt length 2.

We consider the cases where $\omega(L)$ is abelian and non-abelian, and the cases where $L/\omega(L)$ is abelian and non-abelian, giving us four possible cases. There are no Lie algebras in $\mathfrak{W}_2$ which have both $\omega(L)$ and $L/\omega(L)$ non-abelian (see Theorem 5.1.2).

By Theorem 4.6.2, algebras in $\mathfrak{W}_2$ have derived length at most 3. We characterise the algebras which are of derived length exactly 3.

**Lemma 5.1.1** *If $L \in \mathfrak{W}_2$ and $\omega(L)$ is non-abelian, then $L$ is metabelian.*

**Proof** : There are two cases to consider. The first is when $L/\omega(L)$ is abelian. This case is easy, since then

$$L' \leq \omega(L),$$

and we know by Lemma 4.4.2 that

$$L' \cap \omega(L) \leq F(\omega(L))$$

and so $L'$ is abelian since $F(\omega(L))$ is.

The second case is when $L/\omega(L)$ is non-abelian. This case is covered by the next theorem which proves that there is no Lie algebra in $\mathfrak{W}_2$ which has both $\omega(L)$ and $L/\omega(L)$ non-abelian, so this case does not occur. ■

**Theorem 5.1.2** *If $L \in \mathfrak{W}_2$ and $\omega(L)$ is non-abelian, then $L/\omega(L)$ is abelian.*

**Proof** : Suppose that the theorem is false. Let $L \in \mathfrak{W}_2$ be such that both $\omega(L)$ and $L/\omega(L)$ are non-abelian. As a vector space,

$$L = A \oplus B \oplus C \oplus D$$

where $\omega(L) = A \oplus B$, and where, modulo $A \oplus B$, $C \oplus D$ is a non-abelian $T$-algebra ($D$ being the one dimensional space of scalar multiplications).

Let $c \in A \oplus B \oplus C$, $c \notin A \oplus B$, and $0 \neq d \in D$. We prove that $\langle [c,d] \rangle \trianglelefteq L$, which contradicts Corollary 2.4.6, since $\langle [c,d] \rangle \not\leq \omega(L)$.

We can assume, without loss of generality, that $[c,d] = c + x$, where $x \in A \oplus B$. Well, $c + x \in L'$, and $L'$ is nilpotent, thus,

$$\langle c + x \rangle \, si \, L' \trianglelefteq L,$$

so $\langle c + x \rangle \, si \, L$. Therefore, by Lemma 4.4.1, $[a, c + x] = 0$ for all $a \in A$. We also prove that $[y, c + x] = 0$ for all $y \in \omega(L)$.

Suppose that $y \in \omega(L)$. Then, since $\omega(L) \trianglelefteq L$, and by the definition of $\omega(L)$,

$$[y, c + x] \leq \omega(L) \cap \langle c + x \rangle = 0.$$

Therefore, $[\omega(L), c + x] = 0$.

It now suffices to show that $\langle c + x \rangle$ is idealised by all elements of $C \oplus D$. Suppose that $c' \in C$. We show that $[c + x, c'] \in \langle c + x \rangle$.

First suppose that $z \in A \oplus B$. Then

$$\begin{aligned}[c,z] &= [c+x,z]-[x,z]\\ &= -[x,z].\end{aligned}$$

Also suppose that $[c',d] = c'+x'$, where $x' \in A \oplus B$. Let $b \in B$ be such that $[a,b] = a$ for all $a \in A$. Then, by the Jacobi identity,

$$\begin{aligned}0 &= [c,c',b]+[c',b,c]+[b,c,c']\\ &= [c,c']-[x',b,c]+[x,b,c']\\ &= [c,c']+[x',b,x]-[x,b,x']\\ &= [c,c']-[x,x',b],\end{aligned}$$

whence,

$$\begin{aligned}[c,c'] &= [x,x',b]\\ &= [x,x'].\end{aligned}$$

Then,

$$\begin{aligned}[c+x,c'] &= [x,x']+[x,c']\\ &= [x,x']-[x,x']\\ &= 0.\end{aligned}$$

Now suppose that $z \in A \oplus B$. Then,

$$\begin{aligned}0 &= [z,c,d]+[c,d,z]+[d,z,c]\\ &= [x,z,d]+[c+x,z]+[x,[d,z]]\\ &= [x,z,d]+[z,d,x]\\ &= [x,d,z].\end{aligned}$$

Since this is true for all $z \in A \oplus B$, and $[x,d] \in A \oplus B$, $[x,d] \in Z(A \oplus B) = 0$. Thus,

$$\begin{aligned}[c+x,d] &= [c,d]\\ &= c+x.\end{aligned}$$

Therefore, $\langle c+x\rangle \trianglelefteq L$, which contradicts our construction and finishes the proof of the theorem. ∎

We now consider the derived length of algebras in $\mathfrak{W}_2$. The derived length is at most 3. By Lemma 5.1.1, if $\omega(L)$ is non-abelian, then $L$ is metabelian. Thus, if $L \in \mathfrak{W}_2$ and $L$ has derived length exactly 3, $\omega(L)$ is abelian. This clearly implies that $L/\omega(L)$ is non-abelian as otherwise $L$ would be metabelian. We now give an example of a Lie algebra in $\mathfrak{W}_2$ which has derived length exactly 3.

**Example 5.1.3** *Let $L$ be the following Lie algebra: the set $\{d, a_1, a_2, b\}$ is an $F$-basis for $L$, the Lie product is defined by*

$$\begin{aligned} [d,b] &= 2d \\ [a_1,b] &= a_1 \\ [a_2,b] &= a_2 \\ [a_1,a_2] &= d \end{aligned}$$

*and all other products between basis elements are zero. The Lie product is then extended linearly to all of $L$.*

It is straightforward to check that the basis elements of $L$ satisfy the Jacobi identity, so $L$ is a Lie algebra.

We first show that $\omega(L) = \langle d \rangle$. It is straightforward to check that if $x \in \langle d, a_1, a_2 \rangle$ and $0 \neq \delta \in F$, then $(x+\delta b)^L = L$ and so $(x+\delta b)^{sL} = L$. Therefore, $d$ obviously idealises $(x+\delta b)^{sL}$. Also, if $x \in \langle d, a_1, a_2 \rangle$ then $x^{sL} = \langle x \rangle$. Since $d$ idealises $\langle x \rangle$, $\langle d \rangle \leq \omega(L)$. Suppose that $x \in \langle d, a_1, a_2 \rangle$, $\delta \in F$ and $x + \delta b \in \omega(L)$. Then $(d+a_1)^{sL} = \langle d + a_1 \rangle$ and

$$\begin{aligned} [x+\delta b, d+a_1] &= -\delta(2d+a_1) + [x,a_1] \\ &= -\delta(2d+a_1) + \gamma d \quad \text{for some } \gamma \in F. \end{aligned}$$

Since $x + \delta b \in \omega(L)$, $\delta = \gamma$. However, $(2d+a_1)^{sL} = \langle 2d + a_1 \rangle$ and

$$\begin{aligned} [x+\delta b, 2d+a_1] &= -\delta(4d+a_1) + [x,a_1] \\ &= -\delta(4d+a_1) + \gamma d, \end{aligned}$$

which implies that $\gamma = 3\delta$. Thus $\delta = 0$, and $x \in \omega(L)$. Since $\langle d \rangle \leq \omega(L)$ we can assume, without loss of generality, that $x \in \langle a_1, a_2 \rangle$. Let $x = \alpha a_1 + \beta a_2$. Then

$$[x, a_1] = \beta d,$$

which implies $\beta = 0$ since ${a_1}^{sL} = \langle a_1 \rangle$. Also,

$$[x, a_2] = \alpha d,$$

which implies $\alpha = 0$. Therefore, $x = 0$ and $\omega(L) \leq \langle d \rangle$. Thus, $\omega(L) = \langle d \rangle$, and so $L/\omega(L)$ is a non-abelian $T$-algebra. Thus, $L \in \mathfrak{W}_2$. However, $L' = \langle d, a_1, a_2 \rangle$, and then $L^{(2)} = \langle d \rangle$ which is non-zero. However, $L^{(3)}$ is clearly 0 since $L^{(2)}$ is one-dimensional. Therefore, $L$ is a Lie algebra in $\mathfrak{W}_2$ with derived length exactly 3.

We observed above that if $L \in \mathfrak{W}_2$ has derived length 3, then $\omega(L)$ is abelian and $L/\omega(L)$ is non-abelian. In fact, the converse is also true, as proved in the following theorem.

**Theorem 5.1.4** *If $L \in \mathfrak{W}_2$ is such that $L/\omega(L)$ is non-abelian, then $L$ has derived length exactly* 3.

**Proof** : Suppose that the theorem is false. Let $L \in \mathfrak{W}_2$ be a metabelian Lie algebra such that $\omega(L)$ is abelian and $L/\omega(L)$ is non-abelian.

As a vector space, $L = \omega(L) \oplus A \oplus B$, where, modulo $\omega(L)$, $A \oplus B$ is a non-abelian $T$-algebra, with $B$ the one-dimensional algebra of scalar multiplications.

First, we prove $[\omega(L), A] = 0$. Let $a \in A$ and $x \in \omega(L)$. Then, for some $b \in B$, $[a, b] = a + y$ where $y \in \omega(L)$. Thus, $a + y \in L'$. Since $\omega(L)$ is abelian, $F(\omega(L)) = \omega(L)$. Thus, by Lemma 4.4.1, $[a + y, x] = 0$. Thus, $[a, x] = 0$. Therefore, $[\omega(L), A] = 0$.

Next, we prove that $L'' = [A, A]$. Firstly, for all $a \in A$, there is some $y_a \in \omega(L)$, such that $a + y_a \in L'$. Therefore, if $a, a' \in A$, then

$$\begin{aligned} [a, a'] &= [a + y, a' + y_{a'}] \\ &\in L''. \end{aligned}$$

Thus, $[A, A] \leq L''$. However, $L' \leq \omega(L) + A$ and so $L'' \leq [A, A]$. Therefore, $L'' = [A, A]$. However, $L$ is metabelian, thus $[A, A] = 0$.

Suppose that $x \in L \backslash (\omega(L) + A)$. We consider the sub-ideal closure of $x$. Since $[A, A] = 0$, $L' = [\omega(L), B] + [A, B]$. Since $\omega(L) + A$ is abelian, it is straightforward to see that $L' \leq x^L$. However, $\langle x, L' \rangle \trianglelefteq L$, thus $x^L = \langle x, L' \rangle$.

Since $L \in \mathfrak{W}_2$, $x^{sL} = x^{x^L}$. It is clear that $[\omega(L), x] \leq x^{sL}$. Also, $[A, x, x] \leq x^{sL}$. However, all elements of $[A, x]$ are of the form $a + w$, where $a \in A$ and $w \in \omega(L)$. Therefore, all elements of $[A, x, x]$ are of the form $a + w + x$ where $a \in A$, $w \in \omega(L)$ and $x \in [\omega(L), x]$. Since all elements $a + w$ of $[A, x]$ occur in some such element of $[A, x, x]$ and since $[\omega(L), x] \leq x^{sL}$, $[A, x] \leq x^{sL}$. It is also clear that $[\omega(L), x] = [\omega(L), B]$

and that $[A,x] = [A,B]$. Therefore, $L' \leq x^{sL}$. However, any subspace of $L$ which contains $L'$ is clearly an ideal of $L$. Thus, $x^{sL} \trianglelefteq L$. Also, it follows that any subideal containing $x$ must also be an ideal of $L$.

Now consider a subideal $S$ of $L$ such that $S \leq \omega(L) + A$. Suppose that $a \in A$. Then,

$$\begin{aligned} [S,a] &\leq [\omega(L) + A, A] \\ &= 0, \end{aligned}$$

so that $a \in N_L(S)$. Since all other subideals are ideals, $a$ must also normalise them. Thus, we have proved that $a \in \omega(L)$, which implies that $a \in \omega(L) \cap A = 0$. Thus $A = 0$, which contradicts $A \oplus B$ being non-abelian. This contradiction finishes the proof of the theorem. ■

The above Theorem says that the derived length must always be exactly three whenever $L \in \mathfrak{W}_2$ has $L/\omega(L)$ non-abelian. We now characterise the Lie algebras in $\mathfrak{W}_2$ which have derived length exactly 3.

**Theorem 5.1.5** *If $L \in \mathfrak{W}_2$, then $L/\omega(L)$ is non-abelian if and only if $L$ is of the following form; $L$ can be expressed as*

$$L = R \oplus S \oplus A \oplus B$$

*where*

$$B = \langle b \rangle$$

*and*

$$\forall a \in A \ \exists g_a \in R \oplus S \ \ [a,b] = a + g_a \tag{5.1}$$

$$\forall r \in R \ \ [r,b] = 2r \tag{5.2}$$

$$[S,B] \leq R \oplus S \tag{5.3}$$

$$[A,A] \leq R \tag{5.4}$$

$$\forall 0 \neq a \in A \ \exists a' \in A \ \ [a,a'] \neq 0 \tag{5.5}$$

$$b \textit{ acts invertibly on } [R \oplus S, B], \tag{5.6}$$

$$[R \oplus S \oplus A, R \oplus S] = 0 \tag{5.7}$$

**Proof** : Suppose that $L$ is a Lie algebra satisfying conditions 5.1-5.7. It is straightforward to check that the Jacobi identity does not contradict 5.1-5.7, so that there are such Lie algebras.

We prove that $\omega(L) = R \oplus S$. Suppose that $x \in R \oplus S \oplus A$. Then, since $[R \oplus S \oplus A, R \oplus S] = 0$, and since $x^{sL} = \langle x \rangle$, $R \oplus S$ idealises $x^{sL}$. Now suppose that $x \in L \backslash (R \oplus S \oplus A)$. Since $R \oplus S \oplus A$ acts trivially on $R \oplus S$, and $B$ acts invertibly on $[R \oplus S, B]$, it is straightforward to check that $[R \oplus S, B] \leq x^{sL}$. Thus, $R \oplus S$ idealises $x^{sL}$, and $R \oplus S \leq \omega(L)$.

We now prove that $\omega(L) \leq R \oplus S$. It is sufficient to prove that $\omega(L) \cap A \oplus B = 0$, since $\omega(L)$ is a subspace of $L$. First, suppose that $a \in A \cap \omega(L)$. If $a \neq 0$, then let $a' \in A$ be such that $[a, a'] \neq 0$. Then, since $a'^{sL} = \langle a' \rangle$ and $[a, a'] \in R \oplus S$, we have that $[a, a'] \notin (a')^{sL}$, contradicting $a \in \omega(L)$. Thus $a = 0$.

Suppose that $a + b \in \omega(L)$ for some $a \in A$. Then let $0 \neq r \in R$ and $0 \neq a' \in A$. Well,

$$\begin{aligned} [r + a', a + b] &= 2r + [a', a] + [a', b] \\ &= 2r + a' + [a', a] + [a', b] - a'. \end{aligned}$$

Now, $(r + a')^{sL} = \langle r + a' \rangle$, and $[a', a] + [a', b] - a' \in R \oplus S$. Therefore,

$$2r + [a', a] + [a', b] - a' = r,$$

which is to say

$$[a', a] + [a', b] - a' = -r.$$

Replacing $a'$ by $2a'$ in the above argument yields

$$2r + [2a', a] + [2a', b] - 2a' = 2r,$$

which shows that $r = 0$, contradicting our choice of $r$. Thus, $a + b \notin \omega(L)$, which shows that $\omega(L) \leq R \oplus S$.

Therefore, $\omega(L) = R \oplus S$. It is now straightforward to prove that $L \in \mathfrak{W}_2$ has $L/\omega(L)$ non-abelian. This completes half of the proof of the theorem.

Conversely, suppose that $L \in \mathfrak{W}_2$ has $L/\omega(L)$ non-abelian. Suppose that $L = W \oplus A \oplus B$, as a vector space, where $\omega(L) = W$ and where, modulo $W$, $A \oplus B$ is a non-abelian $T$-algebra in the usual way.

We first prove that $[W, A] = 0$. For all $0 \neq a \in A$, there exists $x_a \in W$ such that $a + x_a \in L'$. Suppose that $w \in W$. Then,

$$\begin{aligned} [a, w] &= [a + x_a, w] \\ &= 0, \quad \text{by Lemma 4.4.1.} \end{aligned}$$

Thus, $[W, A] = 0$. We now prove that $L^{(2)} = [A, A]$. Well, $L' \leq W + A$, so it is clear that $L^{(2)} \leq [A, A]$. However, suppose that $a, a' \in A$. Then,

$$[a, a'] = [a + x_a, a' + x_{a'}] \in L^{(2)}.$$

Therefore, $[A, A] = L^{(2)}$.

Define $R = L^{(2)}$. $R$ is spanned by elements of the form $[a, a']$ where $a, a' \in A$. Well,

$$\begin{aligned} 0 &= [a, a', b] + [a', b, a] + [b, a, a'] \\ &= [a, a', b] + [a' + x_{a'}, a] - [a + x_a, a'] \\ &= [a, a', b] - 2[a, a'], \end{aligned}$$

whence,

$$[[a, a'], b] = 2[a, a'],$$

and so for all $r \in R$, $[r, b] = 2r$. Therefore, $L$ satisfies 5.2.

Write $W = R \oplus S$ as a vector space. We calculate $b^{sL}$. Since $L \in \mathfrak{W}_2$, $b^{sL} = b^{b^L}$. Now,

$$b^L = \langle b, R, [S, b], \{a + x_a\}_{a \in A} \rangle.$$

Then it is easy to see that,

$$\begin{aligned} b^{sL} &= b^{b^L} \\ &= \langle b, R, [S, b, b], \{a + x_a + [x_a, b]\}_{a \in A} \rangle. \end{aligned}$$

Now, by the definition of $W$, $[W, b] \in b^{sL}$, so $[W, b] \in b^{sL} \cap W = R + [S, b, b]$. Also, $R = [R, b, b]$, so that $[W, b] \leq [R, b, b] + [S, b, b] = [W, b, b]$. Since $[W, b, b] \leq [W, b]$, $[W, b, b] = [W, b]$. Thus, $L$ satisfies 5.6.

$L$ clearly satisfies 5.1, 5.3, 5.4 and 5.7, so it remains to show that $L$ satisfies 5.5. So, suppose that $0 \neq a \in A$. Then $a \notin \omega(L)$, so there exists some $x \in L$ such that $[x, a] \notin x^{sL}$ First, suppose that $x \notin W \oplus A$. Then, there exists $0 \neq \delta \in F$ and $y \in W$ such that $[a, x] = \delta a + y$. It is clear that $[a, x] \in x^L$. However, $x^{x^L} = x^{sL}$, and $[a, x, x] \in x^{sL}$. But $[a, x, x] = [\delta a + y, x] = \delta^2 a + \delta y + [y, x]$. Now, $y \in \omega(L)$, so $[y, x] \in x^{sL}$, so $\delta^2 a + \delta y \in x^{sL}$, meaning $\delta a + y \in x^{sL}$. This contradiction implies that $x \in W \oplus A$. Therefore, $x^{sL} = \langle x \rangle$. Now, let $x = w + a'$, where $w \in W$ and $a' \in A$. Then

$$\begin{aligned} [a, x] &= [a, w + a'] \\ &= [a, a'] \\ &\neq 0, \end{aligned}$$

since $[a,x] \notin \langle x \rangle$. Therefore, $L$ satisfies 5.5. This completes the proof of the theorem. ∎

**Theorem 5.1.6** *$L \in \mathfrak{W}_2 \backslash \mathfrak{W}_1$ and $\omega(L)$ is non-abelian if and only if $L$ is of the following form.*

$$L = A \oplus B \oplus C,$$

*where $A$ is an $n$-dimensional abelian ideal, $B$ is one-dimensional and acts on $A$ by scalar multiplication and $C$ is non-trivial. The action of $C$ on $A$ is given by the linear transformations, $M_1, M_2 \ldots M_m : A \to A$, defined by*

$$M_j a_i := [a_i, c_j] \quad \text{for all } 1 \le i \le n \tag{5.8}$$

*where $C = \langle c_1, \ldots, c_m \rangle$ and $A = \langle a_1, \ldots, a_n \rangle$. The $M_i$ satisfy the following properties:*

$$M_i M_j = M_j M_i \quad \text{for all } 1 \le i,j \le m \tag{5.9}$$

$$\text{If, for some } \alpha_1, \ldots, \alpha_m \in F, \ \sum_{k=1}^{m} \alpha_k M_k \text{ has an eigenvalue in } F, \text{ then } \alpha_1 = \ldots = \alpha_m = 0 \tag{5.10}$$

*Also,*

$$[C,B] \le A \tag{5.11}$$

$$[c_i, c_j] = [b, c_j, c_i] - [b, c_i, c_j] \quad \text{for all } 1 \le i,j \le m \tag{5.12}$$

**Proof** : We first prove that if $L \in \mathfrak{W}_2 \backslash \mathfrak{W}_1$ and $\omega(L)$ is non-abelian, then it is of the form given in the theorem. $L/\omega(L)$ must be abelian, so $L = A \oplus B \oplus C$ as a vector space, where $\omega(L) = A + B$ and $F(\omega(L)) = A$, and where $C$ is abelian modulo $A + B$.

Let $C = \langle c_1, \ldots, c_m \rangle$. Since $A \trianglelefteq L$, we can define the $M_i$ as in 5.8. Using the Jacobi identity:

$$\begin{aligned} 0 &= [a, c_i, c_j] + [c_j, a, c_i] + [c_i, c_j, a], \quad \text{for any } a \in A, \\ &= [a, c_i, c_j] - [a, c_j, c_i], \quad \text{since } [c_i, c_j] \in A. \text{ Whence,} \\ [a, c_i, c_j] &= [a, c_j, c_i]. \end{aligned}$$

Thus, the $M_i$ commute, so $L$ satisfies 5.9.

Using the Jacobi identity again,

$$\begin{aligned} 0 &= [b, c_i, c_j] + [c_j, b, c_i] + [c_i, c_j, b] \\ &= [b, c_i, c_j] - [b, c_j, c_i] + [c_i, c_j] \quad \text{since } [c_i, c_j] \in A \end{aligned}$$

which gives 5.12.

As in 5.8, we can associate a linear transformation, $M_c$, with all elements $c \in C$. Let $c = \sum_{k=1}^{m} \alpha_k c_k \in C$ be an arbitrary non-zero element of $C$. The linear transformation associated with $c$ is $\sum_{k=1}^{m} \alpha_k M_k$.

We calculate $c^{sL}$. Well, $L' \leq \omega(L)$, so

$$c^L \leq \langle c, A \rangle \trianglelefteq L.$$

Therefore,

$$c^{c^L} \leq c^{\langle c, A \rangle} = \langle c, [c, A] \rangle.$$

$L \in \mathfrak{W}_2$, so all subideals have defect no more than 2. Hence $c^{sL} = c^{c^L}$. $A \leq \omega(L)$, hence $[c, A] \leq c^{sL}$. Then,

$$\langle c, [c, A] \rangle \leq c^{sL} \leq \langle c, [c, A] \rangle,$$

so

$$c^{sL} = \langle c, [c, A] \rangle.$$

Let $a \in A$ and $\delta \in F$. Then, by a similar argument to above,

$$(c - \delta b)^{sL} = \langle c - \delta b, [c - \delta b, A] \rangle,$$

and

$$(c - \delta b + a)^{sL} = \langle c - \delta b + a, [c - \delta b + a, A] \rangle.$$

But,

$$[c - \delta b + a, A] = [c - \delta b, A],$$

since $A$ is abelian. Since $b \in \omega(L)$,

$$[c - \delta b, b], [c - \delta b + a, b] \in [c - \delta b, A].$$

The difference of these two must also be in $[c - \delta b, A]$. Therefore,

$$\begin{aligned} a &= [a, b] \\ &= [c - \delta b + a, b] - [c - \delta b, b] \\ &\in [c - \delta b, A] \end{aligned}$$

Thus $A \leq [c - \delta b, A]$, whence $[c - \delta b, A] = A$. This implies that $\sum_{k=1}^{m} \alpha_k M_k - \delta I$ is invertible for all $\delta \in F$. This is equivalent to saying that $\sum_{k=1}^{m} \alpha_k M_k$ has no eigenvalues in $F$. Thus, $L$ satisfies 5.10. Thus, if $L \in \mathfrak{W}_2 \backslash \mathfrak{W}_1$ has $\omega(L)$ non-abelian, it satisfies 5.8-5.12.

We now prove that if $L$ satisfies 5.8-5.12, then $L \in \mathfrak{W}_2 \backslash \mathfrak{W}_1$ and $\omega(L)$ is non-abelian. Suppose that $L$ has a representation as in the statement of the theorem. It is easy to check that $L$ satisfies the Jacobi identity, so that $L$ is a Lie algebra. We show that $\omega(L) = A + B$. We construct the subideal closure of each element of $L$. If $a \in A$, then $\langle a \rangle$ is a subideal (since $A$ is an abelian ideal), and hence $a^{sL} = \langle a \rangle$. All elements of $A + B$ idealise $\langle a \rangle$. So, let $x$ be an arbitrary element of $(A + B) \backslash A$. Consider $x^{sL}$. Since $A + B \trianglelefteq L$, $x^{sL} \leq A + B$. We calculate $x^L$. It is straightforward to check that $A \leq x^L$, since $[A, x] = A$. Thus, $A \leq x^{sL}$. Since $A = L'$, $x^{sL} \geq \langle A, x \rangle = \langle L', x \rangle \trianglelefteq L$. Therefore, $x^{sL} = \langle A, x \rangle$. Since $x^{sL} \trianglelefteq L$, all elements of $A + B$ idealise $x^{sL}$.

Next, consider an arbitrary element of $L \backslash (A \oplus B)$, $y$. We calculate $y^{sL}$. Note that $y$ acts on $A$ as the linear transformation $\sum_{i=1}^{m} \alpha_i M_i + \delta I$, for some $\alpha_1, \ldots \alpha_m, \delta \in F$. This linear transformation is invertible, otherwise $-\delta$ would be an eigenvalue of $\sum_{i=1}^{m} \alpha_i M_i$, contradicting 5.10.

Thus, as before, $L' = A \leq y^{sL}$, and all elements of $A + B$ idealise $y^{sL}$. Therefore, $A + B \leq \omega(L)$. Suppose that $c \in (C \cap \omega(L)) \backslash \{0\}$. Then, since the linear transformation associated with $c$ has no eigenvalues, $c$ cannot stabilise any one-dimensional subspace of $A$. However, the one-dimensional subspaces of $A$ are all subideals, hence $c \notin \omega(L)$, a contradiction. Since $A + B \leq \omega(L)$, if $x + c \in \omega(L)$, where $x \in A + B$ and $c \in C$, then $c \in \omega(L)$ and hence $c = 0$. Thus, $A + B = \omega(L)$.

Hence, $\omega(L)$ is non-abelian and $L / \omega(L)$ is abelian but non-trivial, implying that $L \in \mathfrak{W}_2 \backslash \mathfrak{W}_1$. This completes the proof of the theorem. ■

This is not a completely satisfactory characterisation, since we have not precisely described the conditions under which there are such linear transformations, or described the form of them. From now on, we fix a basis for $A$ and consider the linear transformations associated with $C$ as the matrices of them with respect to this basis. We now have the following corollary, and will take a detour to describe why characterising these matrices appears so difficult.

**Corollary 5.1.7** *If $F$ is algebraically closed, then $\omega(L)$ is abelian.*

**Proof** : If $F$ is algebraically closed, then there are no matrices without eigenvalues. Thus, by Theorem 5.1.6, there are no Lie algebras in $\mathfrak{W}_2 \backslash \mathfrak{W}_1$ with $\omega(L)$ non-abelian. This completes the proof of the Corollary. ■

Because $\mathbb{R}$ is almost algebraically closed (in the sense that an extension by one element makes it algebraically closed), we can also say much about the structure of these matrices if the underlying field is $\mathbb{R}$.

## 5.2 Those Horrible Matrices

The difficulty in characterising the matrices described in the above section seems to lie in the fact that the existence of such matrices is so intimately linked to the underlying field. A question of eigenvalues of a matrix is about the existence of roots of the characteristic polynomial, and so about the existence of roots of polynomials in $F$. This is why we could so easily dispense with the case where the field is algebraically closed.

Given our field, $F$, and an extension of $F$ of degree $n$, say $\tilde{F}$, we can construct any number, less than $n$, of matrices of the required form. We do this by thinking of our extension field as an $n$-dimensional vector space over $F$. Thus $F$-linear transformations of $\tilde{F}$ can be represented by $n \times n$ matrices over $F$. However, any element, $u$, of $\tilde{F}$ can be associated with a linear transformation, $T_u$ of $\tilde{F}$ by defining $vT_u = vu$ for all $v \in \tilde{F}$. Clearly, the only eigenvalue of $T_u$ over $\tilde{F}$ is $u$, so if $u \notin F$, then $T_u$ has no eigenvalues in $F$. These considerations give rise to the following example, which is conjectured to be typical,

**Example 5.2.1** *Let $F$ be a field of characteristic zero, and $\tilde{F}$ an extension field of $F$ of degree $n$. Let $U$ be an $F$-subspace of $\tilde{F}$ such that $U \cap F = \{0\}$. Let $\{u_1, \ldots, u_m\}$ be an $F$-basis for $U$. Then the matrices associated with $T_{u_1}, \ldots, T_{u_m}$, as above, satisfy the conditions of Theorem 5.1.6.*

We can also prove the following theorem, which strengthens the thought that the above example is typical.

**Theorem 5.2.2** *Let $m$ and $n$ be as in Theorem 5.1.6. Then $m < n$.*

**Proof** : Let $M_1, \ldots, M_m$ be the matrices under consideration, and suppose that they act on $U$, an $n$-dimensional $F$-space with basis $\{u_1, \ldots, u_n\}$.

Then, consider the subspace, $V$, of $M_n(F)$ (all $n \times n$ matrices over $F$) consisting of all matrices which represent a linear transformation, $T$, of $U$ such that $u_1 T = \lambda u_1$ for some $\lambda \in F$. It is clear that $V$ has co-dimension $n-1$ in $M_n(F)$, and that the space spanned by $M_1, \ldots, M_m$ intersects $V$ trivially. This is enough to show that $m \leq n-1$, as required. ■

The above two results leads us to make the following conjecture.

**Conjecture 1 (Peter Neumann)** *Necessary and sufficient conditions for there to be matrices as in Theorem 5.1.6 are that* (1) $m < n$ *and* (2) *that* $F$ *has an extension of degree greater than or equal to* $n$.

The theorem shows that (1) is a necessary condition, and the above example shows that an extension of degree equal to $n$ is a sufficient condition. However I have been unable to prove this conjecture.

## 5.3 Back to characterising $\mathfrak{W}_2$

We now characterise the last of the four possible classes of Lie algebras in $\mathfrak{W}_2 \backslash \mathfrak{W}_1$. We return now to the case where $F$ is an arbitrary field of characteristic $0$. Before characterising the last class from $\mathfrak{W}_2 \backslash \mathfrak{W}_1$, we prove a technical lemma about linear transformations of $F$-spaces.

**Lemma 5.3.1** *Suppose that* $V$ *is a vector space over* $F$, *and that* $P$ *is a space of commuting linear operators on* $V$. *If, for all* $p \in P$,

$$V = Vp \oplus \mathbf{C}_V(p),$$

*is a* $P$*-invariant decomposition of* $V$ *such that* $p$ *is invertible on* $Vp$, *then*

$$V = V_1 \oplus \ldots \oplus V_k,$$

*where, for all* $1 \leq i \leq k$ *and all* $p \in P$, *either* $V_i p = V_i$, *or* $V_i p = 0$.

**Proof** : We prove the lemma by induction on $\dim(V)$. The base case, $\dim(V) = 1$, is trivial.

So suppose that $V$ satisfies the conditions of the Lemma, and that the Lemma is true for all $F$-spaces $U$, where $\dim(U) < \dim(V)$. Let $p \in P$ be arbitrary. We show that $Vp$ and $\mathbf{C}_V(p)$ satisfy the conditions of the lemma.

Then, by induction both of these spaces are decomposable in the required way, so all of $V$ must also be.

So, let $X = Vp$, $Y = \mathbf{C}_V(p)$, and $p' \in P$. We know that $Xp' \leq X$ and $Yp' \leq Y$. We show that $X = Xp' \oplus \mathbf{C}_X(p')$ and $Y = Yp' \oplus \mathbf{C}_Y(p')$ are $P$-invariant decompositions of $X$ and $Y$, respectively.

Suppose that $x \in X$. Then we write $x = x_1 + x_2$ uniquely, where $x_1 \in Vp'$ and $x_2 \in \mathbf{C}_V(p')$. Since $x \in Vp$, there exists $x' \in Vp$ such that $x'p = x$. Then, $x' = x_1' + x_2'$ where $x_1' \in Vp'$ and $x_2' \in \mathbf{C}_V(p')$. Then,

$$\begin{aligned} x_1p' &= xp' \\ &= (x'p)p' \\ &= (x'p')p \\ &= (x_1'p')p \\ &= (x_1'p)p'. \end{aligned}$$

Now, $x_1$ and $x_1'p$ are both elements of $Vp'$, and $p'$ is invertible on $Vp'$. Therefore, $x_1 = x_1'p$. Thus, $x_1 \in X$, by the definition of $X$. Since $X$ is a subspace, we also have $x_2 \in X$. Since $x_1$ and $x_2$ are unique, we have

$$X = Xp' \oplus \mathbf{C}_X(p').$$

It is clear that this is a $P$-invariant decomposition. Since $p'$ acts invertibly on $Vp'$, fixes $X$, as a set, and $Xp' = Vp' \cap X$, $p'$ acts invertibly on $Xp'$.

Now suppose that $y \in Y$. As above, we write $y = y_1 + y_2$, where $y_1 \in Vp'$ and $y_2 \in \mathbf{C}_V(p')$. Now,

$$\begin{aligned} 0 &= (yp')p \\ &= (y_1p')p \\ &= (y_1p)p'. \end{aligned}$$

Since $y_1p \in Vp'$, and $p'$ is invertible on $Vp'$, $y_1p = 0$. Therefore, by the definition of $Y$, $y_1 \in Y$. Hence, since $Y$ is a subspace, $y_2 \in Y$. Again, we have

$$Y = Yp' \oplus \mathbf{C}_Y(p')$$

is a $P$-invariant decomposition. It remains to prove that $p'$ acts invertibly on $Yp'$. But, $Yp' = Y \cap Vp'$ so this is true. This completes the proof of the Lemma. ■

**Theorem 5.3.2** *Let $L$ be a soluble Lie algebra over $F$. Then $L \in \mathfrak{W}_2 \backslash \mathfrak{W}_1$ with $\omega(L)$ and $L/\omega(L)$ both abelian if and only if there are $A$ and $B$ subspaces of $L$ such that $L = A \oplus B$ as a vector space (direct sum) and such that the following properties hold:*

*$A$ is an abelian ideal of $L$ such that,*

$$A = A_1 + \ldots + A_k,$$

*is a direct sum of subspaces of $A$, and $B = \langle b_1, \ldots, b_m \rangle$, for some $m \geq 1$.*

*$[B, B] \leq A$.*

*For all $b \in B \backslash \{0\}$ there is some $x \in L$ such that $[b, x] \neq 0$.*

*For all $1 \leq i \leq k$ and all $b \in B$, $[A_i, b] = A_i$ or $0$.*

*For all $a \in A$, and all $b, b' \in B$, $[a, b, b'] = [a, b', b]$.*

*Either there is some $b \in B \backslash \{0\}$ such that $[A, b] \neq A$ or there is no $b \in B$ such that $b$ acts on $A$ by scalar multiplication.*

**Proof** : We first prove that if $L$ is a Lie algebra with given the structure, then $L \in \mathfrak{W}_2 \backslash \mathfrak{W}_1$ and that $\omega(L)$ and $L/\omega(L)$ are both abelian. We prove that $\omega(L) = A$.

It is clear that $A$ idealises $\langle a \rangle = a^{sL}$ for all $a \in A$, since $A$ is an abelian ideal of $L$. So suppose that $x \in L \backslash A$. It is clear that

$$\begin{aligned} x^L &= \langle x, [L, x] \rangle \\ &= \langle x, [A, x], [B, x] \rangle \quad \text{and,} \\ x^{x^L} &= \langle x, [A, x, x] + [B, x, x] \rangle. \end{aligned}$$

However, $[A, x, x] = [A, x]$ and $[B, x, x] \leq [A, x]$, so $x^{x^L} = \langle x, [A, x] \rangle$. However, since $[A, x, x] = [A, x]$, $[A, x] \leq x^{sL}$, so $x^{sL} = \langle x, [A, x] \rangle$. Thus, for all $x \in L$, $[A, x] \leq x^{sL}$, so $A \leq \omega(L)$.

Suppose $x \in L \backslash A$. Let $x = a + b$, where $a \in A$ and $b \in B$. Then $x \in \omega(L)$ if and only if $b \in \omega(L)$. So we prove that $b \notin \omega(L)$. Suppose that $b$ does not act on $A$ by scalar multiplication. Then, there is some one-dimensional subspace of $A$ (which is a subideal) such that $b$ does not idealise it. Thus, $b \notin \omega(L)$. So suppose that $b$ does act on $A$ by scalar multiplication. Then we know that there is some $b' \in B \backslash \{0\}$ such that $[A, b'] \neq A$. Let $a \in A \backslash [A, b']$. Let $y = a + b' - [b', b]$. Then $y^{sL} = \langle y, [A, b'] \rangle$, since $[b, b'] \in A$. Thus,

$$[y, b] = a + [b', b] - [b', b] = a \notin y^{sL}.$$

Therefore, $b \notin \omega(L)$, so $\omega(L) = A$. It is now clear that $L/\omega(L)$ is non-trivial but abelian so $L \in \mathfrak{W}_2 \backslash \mathfrak{W}_1$, as required.

Now suppose that $L \in \mathfrak{W}_2 \backslash \mathfrak{W}_1$ has both $\omega(L)$ and $L/\omega(L)$ abelian. Then $L = A \oplus B$, where $A = \omega(L)$ is abelian and $B$ is abelian modulo $A$. First, for all $a \in A$ and $b, b' \in B$,

$$\begin{aligned} 0 &= [a,b,b'] + [b,b',a] + [b',a,b] \\ &= [a,b,b'] - [a,b',b], \end{aligned}$$

whence $[a,b,b'] = [a,b',b]$, as required.

Let $b \in B$. We first note that if $b \neq 0$, then $b$ is not in the centre of $L$, so there is some $x \in L$ such that $[x,b] \neq 0$. Now, we calculate $b^{sL}$;

$$\begin{aligned} b^L &= \langle b, [L,b] \rangle \quad \text{and} \\ b^{sL} &= b^{b^L}, \quad \text{since } L \in \mathfrak{W}_2, \\ &= \langle b, [L,b,b] \rangle \\ &\leq \langle b, [A,b] \rangle, \end{aligned}$$

since $[L,b] \leq L' \leq A$. Since $A = \omega(L)$, $[A,b] \leq b^{sL}$, so $b^{sL} = \langle b, [A,b] \rangle$.

Then, $b^{b^{sL}} = \langle b, [A,b,b] \rangle = b^{sL}$, by the definition of $b^{sL}$. Therefore, $[A,b,b] = [A,b]$, so $b$ acts invertibly on $[A,b]$. Let $\{a_1, \ldots, a_n\}$ be a basis for $A$, containing a basis for $[A,b]$, $\{a_1, \ldots, a_r\}$. Then for all $r+1 \leq i \leq n$,

$$[a_i, b] = x_i$$

for some $x_i \in [A,b]$. But, since $[A,b,b] = [A,b]$, there is some $y_i \in [A,b]$ such that $[y_i, b] = x_i$. Let $w_i = a_i - y_i$ for $r+1 \leq i \leq n$. Then it is clear that $[w_i, b] = 0$ and that, $\{a_1, \ldots, a_r, w_{r+1}, \ldots, w_n\}$ is a basis for $A$. We now have

$$\begin{aligned} A &= \langle a_1, \ldots, a_r \rangle \oplus \langle w_{r+1}, \ldots, w_n \rangle \\ &= [A,b] \oplus \mathbf{C}_A(b). \end{aligned}$$

Now suppose that $b' \in B$. Then, if $x \in [A,b]$, then there exists $y \in [A,b]$ such that $[y,b] = x$. Then

$$\begin{aligned} [x,b'] &= [y,b,b'] \\ &= [y,b',b] \\ &\in [A,b]. \end{aligned}$$

Also, if $z \in \mathbf{C}_A(b)$, then

$$\begin{aligned} [z, b', b] &= [z, b, b'] \\ &= 0, \end{aligned}$$

so $[z, b'] \in \mathbf{C}_A(b)$. $A$ and $B$ satisfy the conditions of Lemma 5.3.1. Therefore, there is a decomposition of $A$ into subspaces $A_1, \ldots, A_k$, say, such that all $b$ act either invertibly or trivially on each $A_i$.

It remains to prove that either there is some $b \in B \backslash \{0\}$ or that no $b \in B$ acts on $A$ by scalar multiplication. Suppose that there exists $b \in B$ such that $b$ acts on $A$ by scalar multiplication. Then $b$ stabilises each subideal of $L$ contained in $A$. Thus, there exists $x \in L \backslash A$, such that $[b, x] \notin x^{sL}$, since $b \notin \omega(L)$. We know that $x^{sL} = \langle x, [A, x] \rangle$. If, for each $b' \in B \backslash \{0\}$, $[A, b'] = A$, then $[A, x] = A$. However, in that case $[x, b] \in A \leq x^{sL}$, a contradiction. This completes the proof of the theorem. ∎

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